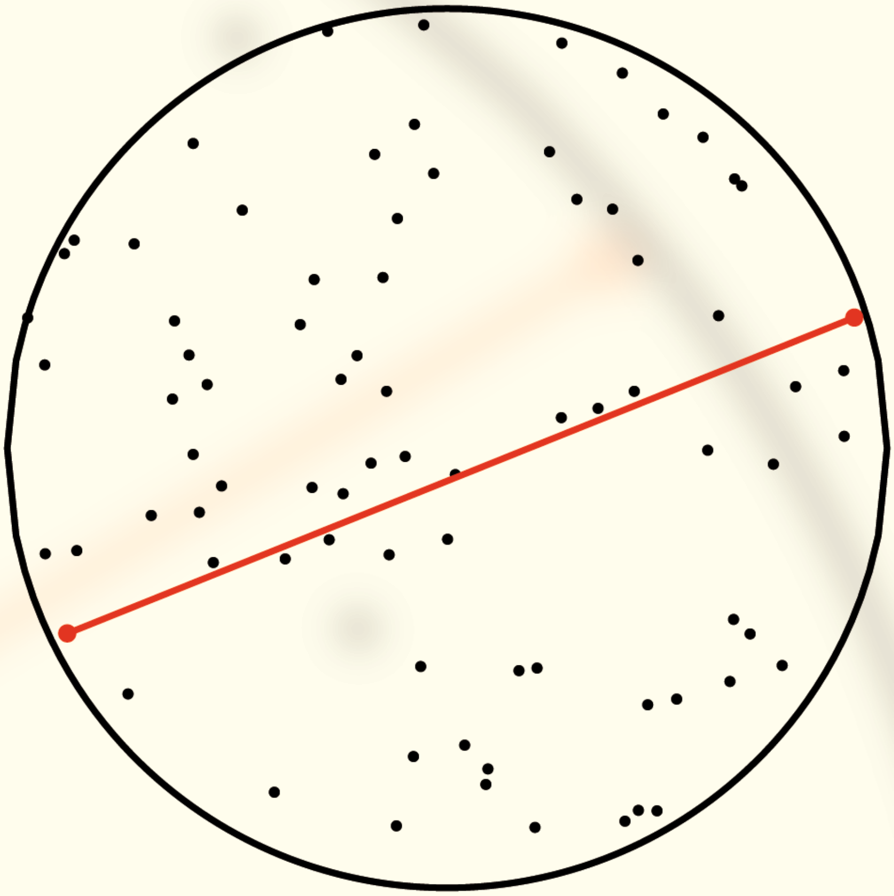
Wei Lao

Some weak limit laws
for the diameter of random point sets in bounded regions



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Dedicated to K.S. Lao, D.H. Ding and Y.M. Xu.

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Wei Lao

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Preface

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Chapter 1

Introduction

We consider a number of random points in some space with some prescribed measure of distance. The largest interpoint distance between two points would seem to be of interest. For example, the maximum distance between pairs of stars describes the diameter of a galaxy, the maximum distance between bullet holes is one of the criteria of the weapon quality, the maximum distance between nodes in a network gives the cover width of the network, the maximum difference between prices of some stock in a time span is a property of its risk class. Alternatively, we may be interested in estimating the diameter of some random set, which is impractical or impossible to be determined directly. Indeed, we can consider the diameter of a random sample from the set as an estimator. In each of these cases, and many others, one may be interested in the asymptotic behavior of the distribution of certain extreme values as the number of points becomes large.

A simple mathematical model for the above situations is as follows. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(X_i)_{i\geq 1}$ a sequence of independent and identically distributed (i.i.d.) random vectors ("points") $X_i : \Omega \to \mathbb{R}^d$, where $d \geq 2$ is a fixed integer. Writing $\|\cdot\|$ for the Euclidean norm in $\mathbb{R}^d$, the largest interpoint distance between $X_1, \ldots, X_n$ is denoted by

$$D_n := \max_{1\leq i<j\leq n} \|X_i - X_j\|.$$

In the terminology of random graphs, the random variable D_n may be considered as a threshold distance for completeness of the random graph $G(\chi_n; r)$ with node set $\chi_n = \{X_1, \ldots, X_n\}$ in which any two points are connected by an edge if their distance is at most r (see Penrose [34], p. 1).

Let $K \subset \mathbb{R}^d$ be the support of the common distribution $\mathbb{P}^{X_1}$ of the random points $X_1, X_2, \ldots$, i.e., K is the smallest closed set such that $\mathbb{P}^{X_1}(K) = 1$. Write

$$\operatorname{diam}(K) := \sup_{x_1, x_2 \in K} \|x_1 - x_2\| \quad (\leq \infty)$$

for the diameter of K. Intuitively, $\operatorname{diam}(K)$ can be approximated by the largest interpoint distance D_n, i.e. the diameter of the random point set $\{X_1, \ldots, X_n\}$. Moreover, for all $\theta < \operatorname{diam}(K)$,

$$\mathbb{P}(D_n \leq \theta) = \mathbb{P}\Big(\bigcap_{\substack{i,j=1,\ldots,n \\ i \neq j}} \{\|X_i - X_j\| \leq \theta\}\Big) \leq \mathbb{P}(\|X_1 - X_2\| \leq \theta)^{\lfloor \frac{n}{2} \rfloor} \longrightarrow 0$$

as $n \to \infty$, where $\lfloor \cdot \rfloor$ denotes the floor function. If $\operatorname{diam}(K) < \infty$, we have

$$\mathbb{P}(D_n \leq \theta) = 1$$

for each $\theta \geq \operatorname{diam}(K)$. Thus $D_n \xrightarrow{P} \operatorname{diam}(K)$ as $n \to \infty$. Since D_n is nondecreasing in n, we conclude that

$$D_n \xrightarrow{\text{a.s.}} \operatorname{diam}(K)$$

as $n \to \infty$. This fact does not provide much information. Our aim throughout this thesis is to provide weak convergence results for D_n, after some suitable centralization and normalization. It is obviously of interest to gain some insight into the speed of convergence of D_n to $\operatorname{diam}(K)$.

In the univariate case $d = 1$ the largest interpoint distance is the sample range, the difference between the sample maximum and the sample minimum. Since the random variables are assumed to be independent, the limit distributions of the maximum and the minimum can be derived by classical extreme value theory, and the asymptotic distribution of the sample range is the convolution of the limit distribution for the extreme order statistics. If, for instance, the distribution of X_i is uniform over [0,1], both $n \cdot (1 - \max_{1 \leq i \leq n} X_i)$ and $n \cdot \min_{1 \leq i \leq n} X_i$ converge in distribution to a standard exponential law (see, e.g., Galambos [17], p. 64-65, or Leadbetter et al. [28], p. 23). Since these two rescaled extreme values are asymptotically independent, it follows that the limit distribution of $n \cdot (1 - D_n)$ is a Gamma distribution with shape parameter 2 and scale parameter 1. Making the transformation $X_i \mapsto 2X_i - 1$, we obtain the limit distribution of the largest interpoint distance for uniformly distributed points in $[-1, 1]$ as follows:

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{n}{2} \cdot (2 - D_n) \leq t\right) = 1 - (1 + t) \cdot e^{-t} \qquad (1.1)$$

for $t > 0$. Another instance is a standard normally distributed sample in $\mathbb{R}$, for which the asymptotic distribution of D_n is well known (see David and Nagaraja [8], p. 211 and exercise 9.3.2), namely, for any $t > 0$,

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left(\sqrt{2\log n}\cdot\left(D_n - 2\sqrt{2\log n} + \frac{\log\log n + \log 4\pi}{\sqrt{2\log n}}\right) \le t\right) \\
&= \int_{-\infty}^{\infty} \exp\left(-u - e^{-t+u} - e^{-u}\right) du.
\end{aligned}
$$

In the multivariate case $d \geq 2$, there are only a few results in the literature for the limit behavior of D_n. The reason is that, in contrast to the case $d = 1$, the largest interpoint distance D_n does not have a simple expression in terms of asymptotically independent extreme order statistics. Moreover, the interpoint distances are based on point pairs, which are not always independent of each other, e.g. (X_1, X_2) and (X_1, X_3) are not independent. Consequently, the classical extreme value theory under the independence condition is not applicable in our context.

If the points are standard normally distributed in $\mathbb{R}^d$, $d \geq 2$, the limit distribution of D_n was obtained by Matthews and Rukhin [30]. Henze and Klein [23] considered the more general case of a multivariate symmetric Kotz distribution $\mathcal{MK}_d(b, \kappa, 1, \mathbf{0}, I_d)$, which contains the standard normal distribution for $b = 1$ and $\kappa = 1/2$. Writing $\mathrm{l}_2 n = \log\log n$ and $\mathrm{l}_3 n = \log \mathrm{l}_2 n$ for short, then for each $t \in \mathbb{R}$,

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\Bigg(\sqrt{(1/\kappa)\log n}\cdot\Bigg(D_n - 2\sqrt{(1/\kappa)\log n} \\
&\qquad\qquad - \frac{(1/2)(d+4b-7)\,\mathrm{l}_2 n + \mathrm{l}_3 n + a}{\sqrt{4\kappa\log n}}\Bigg) \le \frac{t}{2\kappa}\Bigg) \\
&= \exp\left(-e^{-t}\right),
\end{aligned}
$$

where

$$
a = a(d, b) = \log \frac{(d-1)2^{(d-7)/2}\Gamma(d/2)}{\pi^{1/2}\Gamma(d/2 + b - 1)^2}.
$$

There are also some results for uniformly distributed samples. Appel and Russo [5] stated that if the i.i.d. points are uniformly distributed in the d-dimensional unit cube $[0, 1]^d$ associated with the supremum norm $\|\cdot\|_\infty$, then $n\cdot(1-D_n)$ converges in distribution to a non trivial limit law with distribution function $1 - (e^{-t}(t + 1))^d$, $t \geq 0$. In related work, Appel et al. [4] provided a limit theorem for the diameter of uniformly distributed points in a compact planar set A associated with the Euclidean norm. They assumed that A has a unique major axis and that the boundary of A decays strictly faster than the function $\sqrt{x}$ near the endpoints of this major axis. The assumption of sub-$\sqrt{x}$ boundary decay is really restrictive, since many interesting sets are thereby excluded, in particular ellipsoids and balls. However, in case of uniformly distributed points in the unit disc, i.e. $\mathrm{diam}(A) = 2$, Appel et al. [4] gave the rate of the limit distribution of D_n by some bounds, although the existence of the limit distribution is not proven. They stated that

$$\begin{aligned} 1 - \exp\left(-\frac{4t^{5/2}}{3^{5/2}\pi}\right) &\leq \liminf_{n} \mathbb{P}\left(n^{4/5} \cdot (2 - D_n) \leq t\right) \\ &\leq \limsup_{n} \mathbb{P}\left(n^{4/5} \cdot (2 - D_n) \leq t\right) \\ &\leq 1 - \exp\left(-\frac{4t^{5/2}}{\pi}\right) \end{aligned}$$

for each $t > 0$. An exact result for uniformly distributed points in a d-dimensional ball was provided by Lao [26] using a Poisson limit theorem of Silverman and Brown [38]. Independently of Lao, Mayer and Molchanov [32] obtained the same result by first deriving an asymptotic distribution for the diameter of a Poisson point process, and then applying the de-Poissonization technique to obtain the same limit distribution for a general binomial point process. In [32] a more general result was given for spherically symmetric distributed i.i.d. points with certain conditions on the distribution of X_1.

The method in [26] has been used later by Lao and Mayer [27] to derive the limit law of a class of so-called U-max-statistics, which are similar to the well known U-statistics. They considered i.i.d. random elements $X_1, X_2, \ldots$ in some measurable space and a real-valued symmetric measurable function h of k variables. The U-max-statistic of degree k associated with the kernel h is defined by

$$H_n := \max_J h(X_{i_1}, \ldots, X_{i_k}),$$

where the maximum is taken over all permutations $J = \{(i_1, \ldots, i_k) : 1 \leq i_1 < \cdots < i_k \leq n\}$. Notice that the only difference between U-max-statistics and U-statistics is that the former deal with the maximum of the kernel whereas the latter deal with the average of the kernel.

Using some known asymptotic results for U-statistics, Lao and Mayer [27] derived the asymptotic behavior of the distribution of U-max-statistics by establishing some relationship between U-max-statistics and U-statistics. In Chapter 2, we introduce a useful Poisson approximation theorem for some U-statistics. Under the same necessary (and sufficient) conditions on the underlying distribution of the sample, the limit law of U-max-statistics follows from this relationship.

In Chapter 3 we consider a sample of i.i.d. random points in the d-dimensional unit ball. The asymptotic behavior of D_n will be treated for cases with different underlying common distributions that are distinguished by the density of the i.i.d. points near the boundary. The result for the case with a common distribution of power type has been obtained in [27].

Chapter 4 is devoted to the study of the limit distribution of the maximum with three arguments, e.g. the largest surface area or perimeter of all triangles formed by point triplets. We take up the discussion made for uniformly distributed random points on the unit circle.

Chapter 5 contains the limit law of the largest interpoint distance between random points on the edges of a polygon in the unit ball. In this case, we can make use of some known results in the classical extreme theory and some geometric considerations to derive the limit law, instead of using the Poisson limit theorem as before. Simulations can also be found in this chapter to support the obtained result.

In Chapter 6, we turn to the case that the random points are i.i.d. in the support with finite major axes as square or cube. We first derive the limit distribution of the largest interpoint distance between the uniformly distributed points in the unit square. Then, we generalize the method to some other distributions in the unit square, or the uniform distribution in the unit hypercube with dimension $d \geq 2$, or the uniform distribution in a regular convex polygon. We also give some bounds on the limit law of the largest distance between points in an ellipse.

In the last Chapter, we highlight some open problems during the research and give some preliminary considerations.

Chapter 2

Asymptotics for U-max-statistics

As mentioned in Chapter 1, we can define the extremes of symmetric kernels based on random samples as a class of statistics analogous to U-statistics, which we call U-max-statistics. In what follows, we study the asymptotic behavior of the distribution of U-max-statistics as the sample size n tends to infinity.

Let $X_1, X_2, \dots$ be a sequence of random elements in a measurable space $\mathcal{M}$, and let $h : \mathcal{M}^k \to \mathbb{R}$ denote a real-valued measurable function of k variables. Analogous to U-statistics, we may also assume without loss of generality that h is symmetric, because otherwise h could be symmetrized by putting

$$h(x_1, \dots, x_k) = \max_{i_1,\dots,i_k} h(x_{i_1}, \dots, x_{i_k}),$$

where the maximum is taken over all permutations $(i_1, \dots, i_k)$ of $\{1, \dots, k\}$. The U-max-statistic of degree k associated with the kernel h is defined by

$$H_n := \max_{1 \le i_1 < \dots < i_k \le n} h(X_{i_1}, \dots, X_{i_k}),$$

where the maximum is taken over all ordered k-tuples $(i_1, \dots, i_k)$ in $\{1, \dots, n\}$.

The key to derive the limit law of H_n is to construct a U-statistic for each $n \in \mathbb{N}$, which has some relation to H_n and whose limit law can be derived by some known limit theorem. For some $\theta \in \mathbb{R}$ we define the random variable

$$T_n(\theta) := \sum_{1 \le i_1 < \dots < i_k \le n} \mathbf{1}\{h(X_{i_1}, \dots, X_{i_k}) > \theta\}, \tag{2.1}$$

which counts the number of exceedances of the kernel h over the threshold θ. Apart from the factor $\binom{n}{k}^{-1}$, $T_n(\theta)$ is a U-statistic in the usual sense and is equal to zero if and only if H_n does not exceed θ, i.e. we have

$$\{T_n(\theta) = 0\} = \{H_n \le \theta\}.$$

If we can derive the limit distribution of $T_n(\theta)$, then the limit distribution of H_n can also be obtained.

Barbour et al. [6], Theorem 2.M and Corollary 3.D.1, stated a Poisson approximation result for the sum of independent indicator random variables. They gave the exact order of the error between the distribution of the sum and a suitable Poisson distribution by providing upper and lower bounds. Suppose I is a collection of k-subsets $a = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ and, for each $a \in I$, $\mathbf{1}_a$ is an indicator based on $(X_j, j \in a)$, having the value 1 if $(X_j, j \in a)$ satisfies certain condition and the value 0 if not. Denote by W the sum of indicators $W = \sum_{a \in I} \mathbf{1}_a$ and by p_a the expectation of $\mathbf{1}_a$ for each $a \in I$. If $(\mathbf{1}_a, a \in I)$ are independent indicators, then

$$\frac{1}{32}\min(1, \frac{1}{\lambda}) \sum_{a \in I} p_a^2 \le d_{TV}(\mathcal{L}(W), \mathrm{Po}(\lambda)) \le \frac{1 - e^{-\lambda}}{\lambda} \sum_{a \in I} p_a^2,$$

where $\mathcal{L}(W)$ denotes the law of W, $\mathrm{Po}(\lambda)$ stands for the Poisson distribution with parameter $\lambda = \sum_{a \in I} p_a$, and $d_{TV}(\cdot, \cdot)$ is the total variation distance of probability measures.

Notice that, in (2.1) any two indicators $\mathbf{1}\{h(X_{i_1}, \ldots, X_{i_k}) > \theta\}$ and $\mathbf{1}\{h(X_{j_1}, \ldots, X_{j_k}) > \theta\}$ are independent if and only if the two index sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_k\}$ are disjoint. We are thus dealing with dissociated indicator random variables (see [33]); the random number of exceedances $T_n(\theta)$ defined in (2.1) is just a sum of dissociated indicators. Barbour et al. [6] also generalized their Theorem 2.M to cover the case of dissociated random indicators. Using the notations above, the family $(\mathbf{1}_a, a \in I)$ is said to be dissociated if for each $A, B \subset I$ the subsets of random variables $(\mathbf{1}_a, a \in A)$ and $(\mathbf{1}_b, b \in B)$ are independent whenever $(\bigcup_{a \in A} a) \cap (\bigcup_{b \in B} b) = \emptyset$. In the case $k = 1$, a dissociated family of indicators is equivalent to an independent family. If $k \ge 2$, there is a much wider scope. For each a, define $I_a^s = \{b \in I : b \neq a, b \cap a \neq \emptyset\}$. Barbour et al. [6], Theorem 2.N, gave an upper bound of the error between the law of the sum W of dissociated indicators and a Poisson distribution with parameter $\lambda = \sum_{a \in I} p_a$ as follows:

$$d_{TV}(\mathcal{L}(W), \mathrm{Po}(\lambda)) \le \lambda^{-1}(1 - e^{-\lambda}) \sum_{a \in I} \Big(p_a^2 + \sum_{b \in I_a^s} (p_a p_b + \mathbb{E}(\mathbf{1}_a \mathbf{1}_b)) \Big).$$

In our case where I is the class of all k-subsets of $\{1, \ldots, n\}$ and the random variables $X_1, \ldots, X_n$ are i.i.d., the upper bound above can be given more explicitly. The following result can also be found in Barbour et al. [6], p. 35, which is a direct consequence of the inequality above.

Theorem 2.1. *For any $\theta \in \mathbb{R}$ and $n \geq k$, let*

$$\begin{aligned} p(\theta) &:= \mathbb{P}\left(h(X_1, \ldots, X_k) > \theta\right), \\ \lambda_n(\theta) &:= \mathbb{E}\left(T_n(\theta)\right) = \binom{n}{k} p(\theta), \end{aligned}$$

and, for $r = 1, \ldots, k-1$,

$$\tau_r(\theta) := p(\theta)^{-1} \mathbb{P}\left(h(X_1, \ldots, X_k) > \theta, h(X_{1+k-r}, X_{2+k-r}, \ldots, X_{2k-r}) > \theta\right).$$

We then have

$$\begin{aligned} & d_{TV}\left(\mathcal{L}(T_n(\theta)), \mathrm{Po}(\lambda_n(\theta))\right) \\ \leq\ & (1 - e^{-\lambda(\theta)}) \left(p(\theta) \left[\binom{n}{k} - \binom{n-k}{k} \right] + \sum_{r=1}^{k-1} \binom{k}{r} \binom{n-k}{k-r} \tau_r(\theta) \right), \end{aligned} \tag{2.2}$$

where the sum is defined to be zero if $k = 1$.

Note that $p(\theta)$ is the common value of the expectations of the indicators $\mathbf{1}_a = \mathbf{1}\{h(X_i, i \in a) > \theta\}$, $a \in I$, and $\tau_r(\theta)$ is the common value of $p(\theta)^{-1}\mathbb{E}\left(\mathbf{1}_a \mathbf{1}_b\right) = \mathbb{P}\left(\mathbf{1}_b = 1 \mid \mathbf{1}_a = 1\right)$ for all pairs $a, b \in I$ for which $|a \cap b| = r$. Obviously, the behavior of the upper tail of the distribution of the kernel h plays an important role.

One of the main applications of this theorem is that the law of the number of exceedances $T_n(\theta)$ converges to a Poisson distribution as $n \to \infty$ if the expectation $\lambda_n(\theta)$ converges to some positive constant for each θ and the upper bound of the error converges to zero as $n \to \infty$. We then get an approximation of the law of the U-max-statistic H_n as $n \to \infty$. It is obvious that for a fixed threshold θ the expectation $\lambda_n(\theta)$ either equals zero for all n or converges to infinity as $n \to \infty$. We therefore must find a suitable sequence of transformations $\theta_n : \Theta \to \mathbb{R}$ with $\Theta \subset \mathbb{R}$ such that the tail probability $p(\theta_n(t))$, $t \in \Theta$, decreases in n and both the convergence of $\lambda_n(\theta_n(t))$ to some positive value and the convergence of the upper bound of the error to zero hold for each $t \in \Theta$. Without loss of generality, we can choose $\Theta = [0, \infty)$.

Poisson approximation in this context was first considered by Silverman and Brown [38], who considered the statistical analysis of point patterns and stated that

$$p(\theta_n(t)) \leq \tau_1(\theta_n(t)) \leq \cdots \leq \tau_{k-1}(\theta_n(t)) \leq 1 \tag{2.3}$$

for each $n \geq k$ and each $t \in \Theta$. On the other hand, since $\binom{m}{l} = O(m^l)$ for fixed l and $m \to \infty$, the upper bound in (2.2) is of the order

$$O\left(p(\theta_n(t))n^{k-1} + \sum_{r=1}^{k-1} \tau_r(\theta_n(t))n^{k-r}\right)$$

as $n \to \infty$. Moreover, the upper bound of the error converges to zero if and only if $p(\theta_n(t))n^{k-1} \to 0$ and the summands converge to zero as $n \to \infty$. Suppose that $\lambda_n(\theta_n(t))$ converges to some positive constant, which means that $p(\theta_n(t))$ must be of the order $O(n^{-k})$ and thus $p(\theta_n(t))n^{k-1}$ converges to zero. It thus remains to prove that the summands converge to zero. Using the inequalities (2.3), we obtain a simplified corollary, which was stated by Silverman and Brown [38].

Corollary 2.2. *For $\Theta \subset \mathbb{R}$, let $(\theta_n)_{n\geq k}$ be a sequence of transformations $\theta_n : \Theta \to \mathbb{R}$. Suppose that for each $t \in \Theta$ there is a constant $\lambda(t) \in (0, \infty)$ such that*

$$\lim_{n\to\infty} \binom{n}{k} \mathbb{P}\left(h(X_1, \ldots, X_k) > \theta_n(t)\right) = \lambda(t) \tag{2.4}$$

and

$$\lim_{n\to\infty} n^{2k-1} \mathbb{P}\left(h(X_1, \ldots, X_k) > \theta_n(t), h(X_1, \ldots, X_{k-1}, X_{k+1}) > \theta_n(t)\right) = 0. \tag{2.5}$$

We then have

$$T_n(\theta_n(t)) \xrightarrow{\mathcal{D}} \mathrm{Po}(\lambda(t)),$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

Using this Poisson approximation on the special set $\{T_n(\theta_n(t)) = 0\} = \{H_n \leq \theta_n(t)\}$, we obtain the limit behavior of the law of H_n:

$$\lim_{n\to\infty} \mathbb{P}\left(H_n \leq \theta_n(t)\right) = \exp\left(-\lambda(t)\right), \quad t \in \Theta.$$

Hence, this is the main tool to derive the asymptotic results in Chapter 3.

Note that (2.4) is equivalent to the convergence

$$\lambda_n(\theta_n(t)) = \mathbb{E}\left(T_n(\theta_n(t))\right) \longrightarrow \lambda(t)$$

as $n \to \infty$, which implies tightness of the sequence $(T_n)_{n\geq k}$, a necessary criterion for weak convergence of distributions. Furthermore, we can calculate the variance of T_n as follows:

$$
\begin{aligned}
&\mathbb{V}\,(T_n(\theta_n(t))) \\
=&\sum_{a\in I}\mathbb{V}\,(\mathbf{1}_a) + \sum_{a,b\in I}\mathbb{C}\mathrm{ov}\,(\mathbf{1}_a,\mathbf{1}_b) \\
=&\binom{n}{k} p(\theta_n(t))[1-p(\theta_n(t))] + \sum_{a\cap b\neq\emptyset}\left(\mathbb{E}\,(\mathbf{1}_a\mathbf{1}_b) - p(\theta_n(t))^2\right).
\end{aligned}
$$

Here, the last sum is over all distinct pairs $a, b \in I$ such that $a \cap b \neq \emptyset$. Under the first condition, the sum of the variances of indicators converges to the constant $\lambda(t)$, and the sum of the covariances of indicators is equal to

$$
\begin{aligned}
&\sum_{r=1}^{k-1}\binom{n}{k}\binom{k}{r}\binom{n-r}{k-r}\left(\mathbb{P}\,(\mathbf{1}_a = 1, \mathbf{1}_b = 1) - p(\theta_n(t))^2\right) \\
=\;& O\left(\sum_{r=1}^{k-1} n^{2k-r} p(\theta_n(t)) \cdot \tau_r(\theta_n(t))\right).
\end{aligned}
\tag{2.6}
$$

From condition (2.5) and inequality (2.3) we conclude that each of the summands in (2.6) converges to zero, which implies

$$
\lim_{n\to\infty}\mathbb{V}\,(T_n(\theta_n(t))) = \lambda(t). \tag{2.7}
$$

Obviously, condition (2.5) is sufficient but not necessary for (2.7). In the case $k > 2$, it is sometimes useful to replace this single condition by the weaker conditions

$$
\lim_{n\to\infty} n^{2k-r} p(\theta_n(t))\tau_r(\theta_n(t)) = 0
$$

for $r = 1, \ldots, k-1$.

Chapter 3

Largest distance in a ball

Using the results of Chapter 2, this chapter deals with the limit distribution of the largest interpoint distance between points in the unit ball. The first two sections are preparations for the discussion. Section 3.3 states the results given in Lao and Mayer [27], who considered the case that the underlying point distribution belongs to the so-called *power type*. Some of the asymptotic considerations in the proofs of Lao and Mayer are studied in more detail. In Section 3.4 we turn to another class of point distributions, called of *logarithmic type*, where the points lie more likely near the boundary of the unit ball than in the case of power type. The opposite case is the *exponential type* point distributions discussed in Section 3.5, where we shall see that the second condition of Corollary 2.2 is not satisfied. However, we detect an interesting phenomenon. In the last section we shall deal with some special cases where the support of the point distributions is a proper subset of the unit ball.

3.1 Preliminaries

Let $(X_i)_{i\geq 1}$, $X_i : \Omega \to \mathbb{R}^d$, be a sequence of i.i.d. random points, where $d \geq 2$. As before, write

$$D_n := \max_{1\leq i<j\leq n} \|X_i - X_j\|$$

for the largest interpoint distance between $X_1, \ldots, X_n$.

This chapter studies the limit distribution of D_n under some general conditions where the support of $\mathbb{P}^{X_1}$ is some ball in $\mathbb{R}^d$ (except in the last section) which, because of translation invariance and the fact that D_n is scale equivariant (i.e., $D_n(tX_1, \ldots, tX_n) = t \cdot D_n(X_1, \ldots, X_n)$, $t > 0$), may be taken without loss of generality to be the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$. A notable exception can be found in Section 3.6 where we discuss some special cases in which the support of $\mathbb{P}^{X_1}$ is contained in the unit ball and $\mathbb{P}^{X_1}(A) = 0$ for some subset $A \subset \mathbb{B}^d$ of positive Lebesgue measure.

If the support of $\mathbb{P}^{X_1}$ is $\mathbb{B}^d$, the random variable D_n converges almost surely to $\operatorname{diam}(\mathbb{B}^d) = 2$ as $n \to \infty$. To study the asymptotic behavior of $2 - D_n$, we choose a sequence of thresholds $2 - \varepsilon_n$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and write

$$T_n := \sum_{1 \le i < j \le n} \mathbf{1}\{\|X_i - X_j\| > 2 - \varepsilon_n\}$$

for the number of exceedances of the interpoint distances over this threshold. By Corollary 2.2, if conditions (2.4) and (2.5) hold, the random variable T_n converges in distribution to a Poisson distribution as $n \to \infty$. As a consequence, we have the following lemma.

Lemma 3.1. *Let $X_1, X_2, \ldots$ be i.i.d. points in $\mathbb{B}^d$ and, for fixed $t > 0$, let $(\varepsilon_n(t))_{n \in \mathbb{N}}$ be a sequence of positive real numbers satisfying $\lim_{n \to \infty} \varepsilon_n(t) = 0$. If*

$$\lim_{n \to \infty} \binom{n}{2} \mathbb{P}(\|X_1 - X_2\| > 2 - \varepsilon_n(t)) = \lambda(t) \tag{3.1}$$

for some $\lambda(t) \in (0, \infty)$ and

$$\lim_{n \to \infty} n^3 \mathbb{P}(\|X_1 - X_2\| > 2 - \varepsilon_n(t), \|X_1 - X_3\| > 2 - \varepsilon_n(t)) = 0, \tag{3.2}$$

then

$$\lim_{n \to \infty} \mathbb{P}(2 - D_n \le \varepsilon_n(t)) = 1 - \exp(-\lambda(t)).$$

Suppose throughout this chapter that $\mathbb{P}(X_1 = 0) = 0$. For each random point X_i, $i = 1, 2, \ldots$, write

$$R_i(\omega) := \|X_i(\omega)\|, \quad \omega \in \Omega$$

for the random distance (radius) between the point and the origin, and put

$$U_i(\omega) := \begin{cases} \frac{X_i(\omega)}{\|X_i(\omega)\|}, & \text{if } X_i(\omega) \neq 0, \\ (1, 0, \ldots, 0)^T, & \text{otherwise.} \end{cases}$$

Then, $U_i(\omega)$ is a point (angle) on the surface $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ of the unit ball which describes the direction of the point X_i. Since $\mathbb{P}(X_1 = 0) = 0$, there is a subset $\Omega_0 \in \mathcal{A}$ such that $\mathbb{P}(\Omega_0) = 1$ and $X_i(\omega) = R_i(\omega) \cdot U_i(\omega)$, $i = 1, 2, \ldots$, for each $\omega \in \Omega_0$. We assume throughout this chapter that R_i and U_i are independent for $i = 1, 2, \ldots$. This condition holds if X_1 has a spherical symmetric distribution. In this case the variables U_i, $i = 1, 2, \ldots$ are independent and uniformly distributed on $\mathbb{S}^{d-1}$.

As is common in classical extreme value theory, we classify the distribution of the radius by its tail behavior. To this end, write $t_F := \sup\{x \in \mathbb{R} : F(x) < 1\}$ for the right endpoint of a univariate distribution function F. The following concept defines an equivalence relation on the set of all distribution functions.

Definition 3.1. Two distribution functions F_1 and F_2 are called *tail-equivalent* if $t_{F_1} = t_{F_2}$, and if

$$\lim_{t \uparrow t_{F_1}} \frac{1 - F_1(t)}{1 - F_2(t)} = \lim_{s \downarrow 0} \frac{1 - F_1(t_{F_1} - s)}{1 - F_2(t_{F_1} - s)} = c$$

for some constant $c \in (0, \infty)$.

We also need the following concept of asymptotic equivalence of functions.

Definition 3.2. Let ψ_1, ψ_2 be real-valued functions defined on some nondegenerate interval $M \subset \mathbb{R}$, and let $s_0 \in M$. If $\psi_2(s)$ is non-zero for each s sufficiently close to s_0, we write $\psi_1(s) \sim \psi_2(s)$ as $s \to s_0$ if and only if

$$\lim_{s \to s_0} \frac{\psi_1(s)}{\psi_2(s)} = 1.$$

In this case ψ_1 and ψ_2 are called *asymptotically equivalent* as $s \to s_0$.

According to this definition, the concept of tail-equivalence means that

$$1 - F_1(t_F - s) \sim c \cdot (1 - F_2(t_F - s))$$

for some $c \in (0, \infty)$ as $s \downarrow 0$. The following lemmas are also useful for later purposes.

Lemma 3.2. *Let $\psi_1, \psi_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be measurable integrable positive real-valued functions such that $\psi_1(s) \sim \psi_2(s)$ as $s \downarrow 0$ and, let $h : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function such that, for each $s > 0$, $h(s, \cdot)$ is positive and integrable on the interval* $(0, s)$. *We then have*

(a) $$\int_0^s \psi_1(t) \cdot h(s,t)\, dt \sim \int_0^s \psi_2(t) \cdot h(s,t)\, dt$$

(b) $$\int_0^s \psi_1(s-t) \cdot h(s,t)\, dt \sim \int_0^s \psi_2(s-t) \cdot h(s,t)\, dt$$

as $s \to 0$. In particular, if $h(s,t) = 1$ for all $s, t \in \mathbb{R}$, we have

$$\int_0^s \psi_1(t)\, dt \sim \int_0^s \psi_2(t)\, dt \quad \text{and} \quad \int_0^s \psi_1(s-t)\, dt \sim \int_0^s \psi_2(s-t)\, dt$$

as $s \to 0$.

Lemma 3.3. *Let F be a distribution function satisfying $F(0) = 0$ and Ψ a function defined on the support of F with $\Psi(0) = 0$. Suppose that F and Ψ are differentiable and $\Psi' > 0$ in a right neighborhood of 0. Moreover, assume $F'(s) \sim \Psi'(s)$ as $s \to 0$. We then have*

(a) $F(s) \sim \Psi(s)$,

(b) $$F * F(s) \sim \int_0^s \Psi(s-x)\, \Psi'(x)\, dx$$

as $s \to 0$, where "$*$" denotes convolution.

The proofs of these two lemmas can be found in Appendix A.

Throughout this chapter, $\measuredangle(\cdot, \cdot)$ denotes the central angle between two points. (The central angle means here the smaller of the two angles at the origin, it does not mean the reflex angle.) The s-dimensional Hausdorff measure on $\mathbb{R}^n$ (see Appendix B.1) will be denoted by $\mathscr{H}^s$. Note that the restriction of the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$ to $\mathbb{S}^{d-1}$ is the surface area measure, which is denoted by μ^{d-1}. In the computation we also use the Gamma function and the Beta function, which are denoted by $\Gamma(\cdot)$ and $\mathrm{B}(\cdot, \cdot)$, respectively.

3.2 Tail probability

We already know that the tail behavior of the distance between two random points is of crucial importance for the limit behavior of D_n. In the following, we derive some bounds on the probability that the distance between two points exceeds a threshold which is close to 2. We then give the asymptotic behavior of this probability under some further conditions on the point distribution.

Let X_1, X_2 be i.i.d. points in $\mathbb{B}^d$, $d \geq 2$, such that $X_i = R_i U_i$, $i = 1, 2$, where $R_i = \|X_i\|$ and $U_i = X_i / \|X_i\| \in \mathbb{S}^{d-1}$ are independent. Suppose that U_i has a density g with respect to μ^{d-1}. Denote by $\phi := \measuredangle(-U_1, U_2)$ the central angle between $-U_1$ and U_2. By using the law of cosines we have for each $0 < \varepsilon < 2$

$$\begin{aligned}
&\mathbb{P}(\|X_1 - X_2\| \geq 2 - \varepsilon) \\
= \; &\mathbb{P}\Big(R_1^2 + R_2^2 + 2R_1R_2\cos\phi \geq (2-\varepsilon)^2,\ R_1 + R_2 \geq 2 - \varepsilon\Big) \\
= \; &\mathbb{P}\left(\cos\phi \geq \frac{(2-\varepsilon)^2 - R_1^2 - R_2^2}{2R_1R_2},\ R_1 + R_2 \geq 2 - \varepsilon\right).
\end{aligned}$$

For small ε, an exceedance can only happen for a very small random angle ϕ. The second order Taylor approximation to $\cos\phi$ around 0 is $1 - \frac{1}{2}\phi^2\cos\xi$, where $\xi \in [0, \phi]$. Using this approximation and plugging $Y_i := 1 - R_i$, $i = 1, 2$, into the last expression, we get

$$\begin{aligned}
&\mathbb{P}(\|X_1 - X_2\| \geq 2 - \varepsilon) \\
= \; &\mathbb{P}\left(1 - \frac{1}{2}\phi^2\cos\xi \geq \frac{(2-\varepsilon)^2 - (1-Y_1)^2 - (1-Y_2)^2}{2(1-Y_1)(1-Y_2)},\ Y_1 + Y_2 \leq \varepsilon\right) \\
= \; &\mathbb{P}\left(\frac{1}{2}\phi^2\cos\xi \leq \frac{4(\varepsilon - Y_1 - Y_2) + (Y_1 + Y_2)^2 - \varepsilon^2}{2(1-Y_1)(1-Y_2)},\ Y_1 + Y_2 \leq \varepsilon\right) \\
= \; &\mathbb{P}\Big(\phi^2\cos\xi \leq 4(\varepsilon - Y_1 - Y_2) + \varrho(\varepsilon, Y_1, Y_2),\ Y_1 + Y_2 \leq \varepsilon\Big) \qquad (3.3) \\
=: \; &p(\varepsilon),
\end{aligned}$$

where

$$\varrho(\varepsilon, Y_1, Y_2) := \frac{4(\varepsilon - Y_1 - Y_2)(Y_1 + Y_2 - Y_1Y_2) + (Y_1 + Y_2)^2 - \varepsilon^2}{(1-Y_1)(1-Y_2)}. \qquad (3.4)$$

Since $Y_1 + Y_2 \leq \varepsilon$ implies

$$\begin{aligned}
|\varrho(\varepsilon, Y_1, Y_2)| &\leq \frac{4\cdot|\varepsilon - Y_1 - Y_2|\cdot|Y_1 + Y_2 - Y_1Y_2| + (Y_1 + Y_2)^2 + \varepsilon^2}{|1 - Y_1|\cdot|1 - Y_2|} \\
&\leq \frac{6\varepsilon^2}{(1-\varepsilon)^2},
\end{aligned}$$

and $6\varepsilon^2/(1-\varepsilon)^2 = 6\varepsilon^2 + O\big(\varepsilon^3\big)$ by Taylor series expansion, we obtain the inequality $|\varrho(\varepsilon, Y_1, Y_2)| \leq 7\varepsilon^2$ for sufficiently small ε.

Moreover, by some geometric considerations the upper bound on ϕ for the distance exceeding the threshold $2 - \varepsilon$ reaches its maximum at $R_1 = R_2 = 1$, and it follows from the inequality $\|X_1 - X_2\| \geq 2 - \varepsilon$ that

$$\cos\phi \geq \frac{(2-\varepsilon)^2 - 2}{2} = 1 - 2\varepsilon + \frac{1}{2}\varepsilon^2.$$

Since $0 \le \xi \le \phi$ and the cosine function is decreasing on the interval $[0, \pi]$, we have

$$1 \ge \cos\xi \ge \cos\phi \ge 1 - 2\varepsilon + \frac{1}{2}\varepsilon^2,$$

where the lower bound is positive for sufficiently small ε. Moreover, for the reciprocal a Taylor series expansion yields

$$1 \le \frac{1}{\cos\xi} \le \frac{1}{1 - 2\varepsilon + \frac{1}{2}\varepsilon^2} = 1 + 2\varepsilon + O\left(\varepsilon^2\right),$$

whence $1 \le \frac{1}{\cos\xi} \le 1 + 3\varepsilon$ for sufficiently small ε.

We now derive inequalities for $p(\varepsilon)$ by putting the bounds on $\varrho(\varepsilon, Y_1, Y_2)$ and on $1/\cos\xi$ into (3.3). On one hand, we have for sufficiently small ε

$$\begin{aligned} p(\varepsilon) &\ge \mathbb{P}\left(\phi^2 \le 4(\varepsilon - Y_1 - Y_2) - 7\varepsilon^2,\ Y_1 + Y_2 \le \varepsilon\right) \\ &= \mathbb{P}\left(\phi^2 \le 4(\varepsilon - Z) - 7\varepsilon^2,\ Z \le \varepsilon\right) \\ &= \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \mathbb{P}\left(\phi^2 \le 4(\varepsilon - z) - 7\varepsilon^2\right)\, d\mathbb{P}^Z(z) \\ &= \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \mathbb{P}\left(\phi \le \left(4(\varepsilon - z) - 7\varepsilon^2\right)^{1/2}\right)\, d\mathbb{P}^Z(z) \\ &=: p_1(\varepsilon), \end{aligned}$$

where $Z := Y_1 + Y_2$ and $\mathbb{P}^Z$ is the convolution of $\mathbb{P}^{Y_1}$ and $\mathbb{P}^{Y_2}$. On the other hand, we have

$$\begin{aligned} p(\varepsilon) &= \mathbb{P}\left(\phi^2 \le \frac{1}{\cos\xi} \cdot (4(\varepsilon - Y_1 - Y_2) + \varrho(\varepsilon, Y_1, Y_2)),\ Y_1 + Y_2 \le \varepsilon\right) \\ &\le \mathbb{P}\left(\phi^2 \le (1 + 3\varepsilon) \cdot (4(\varepsilon - Y_1 - Y_2) + 7\varepsilon^2),\ Y_1 + Y_2 \le \varepsilon\right) \\ &\le \mathbb{P}\left(\phi^2 \le 4(\varepsilon - Y_1 - Y_2) + 20\varepsilon^2,\ Y_1 + Y_2 \le \varepsilon\right) \\ &= \mathbb{P}\left(\phi^2 \le 4(\varepsilon - Z) + 20\varepsilon^2,\ Z \le \varepsilon\right) \\ &= \int_{z=0}^{\varepsilon} \mathbb{P}\left(\phi^2 \le 4(\varepsilon - z) + 20\varepsilon^2\right)\, d\mathbb{P}^Z(z) \\ &\le \int_{z=0}^{\varepsilon + 5\varepsilon^2} \mathbb{P}\left(\phi \le \left(4(\varepsilon - z) + 20\varepsilon^2\right)^{1/2}\right)\, d\mathbb{P}^Z(z) \\ &=: p_2(\varepsilon). \end{aligned}$$

Like the lower bound $p_1(\varepsilon)$, this upper bound is only valid for sufficiently small ε. For example, the second inequality involves the estimate $21\varepsilon^3 \le \varepsilon^2$, which holds if $\varepsilon \le 1/21$.

Both the integrands in $p_1(\varepsilon)$ and in $p_2(\varepsilon)$ involve probabilities of the form $\mathbb{P}\left(\phi \le \sqrt{\eta}\right)$ for small η. With help of some geometric considerations, we can prove the following result.

Lemma 3.4. *Let U_1, U_2 be i.i.d. points on $\mathbb{S}^{d-1}$, $d \ge 2$, with bounded density g with respect to μ^{d-1} and let $\phi = \measuredangle(-U_1, U_2)$. We then have*

$$\mathbb{P}(\phi \le \sqrt{\eta}) \sim \beta \cdot \eta^{\frac{d-1}{2}}$$

as $\eta \to 0$, where

$$\beta = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \int_{\mathbb{S}^{d-1}} g(u)g(-u)\mu^{d-1}(du). \tag{3.5}$$

Proof. For fixed $u \in \mathbb{S}^{d-1}$, let $\phi_u := \measuredangle(-u, U_2)$ be the central angle between $-u$ and U_2. We then have

$$\mathbb{P}(\phi \le \sqrt{\eta}) = \int_{\mathbb{S}^{d-1}} \mathbb{P}(\phi_u \le \sqrt{\eta})\, d\mathbb{P}^{U_1}(u).$$

Write

$$C_u(\eta) := \left\{ v \in \mathbb{S}^{d-1} : \measuredangle(-u, v) \le \sqrt{\eta} \right\}$$

for the spherical cap that contains the points on $\mathbb{S}^{d-1}$ with a central angle to $-u$ of at most $\sqrt{\eta}$ (see Figure 3.1). By a general form of Lebesgue's differentiation theorem (see Appendix A) we obtain

$$\lim_{\eta \to 0} \frac{\mathbb{P}\left(\phi_u \le \sqrt{\eta}\right)}{\mu^{d-1}(C_u(\eta))} = g(-u)$$

for μ^{d-1}-almost every $u \in \mathbb{S}^{d-1}$. Furthermore, since

$$\mathbb{P}(\phi_u \le \sqrt{\eta}) \le \sup_{u \in \mathbb{S}^{d-1}} g(u) \cdot \mu^{d-1}(C_u(\eta))$$

and g is a bounded function, $\left|\mathbb{P}\left(\phi_u \le \sqrt{\eta}\right)/\mu^{d-1}(C_u(\eta))\right|$ is bounded by a positive constant and hence integrable with respect to any finite measure. By the

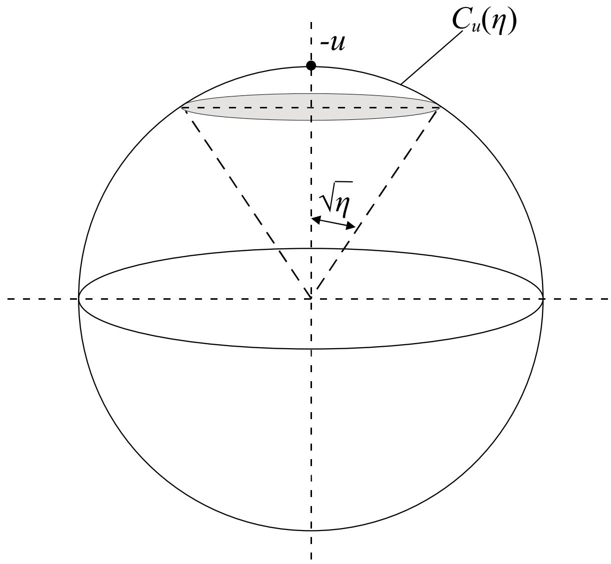

Figure 3.1: The illustration of the geometric consideration.

dominated convergence theorem (see Appendix A), integration with respect to the angular distribution yields

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} g(-u)\, d\mathbb{P}^{U_1}(u) &= \lim_{\eta \to 0} \int_{\mathbb{S}^{d-1}} \frac{\mathbb{P}\left(\|u+\phi\| \le \sqrt{\eta}\right)}{\mu^{d-1}(C_u(\eta))}\, d\mathbb{P}^{U_1}(u) \\
&= \lim_{\eta \to 0} \frac{1}{\mu^{d-1}(C_u(\eta))} \int_{\mathbb{S}^{d-1}} \mathbb{P}\left(\|u+\phi\| \le \sqrt{\eta}\right)\, d\mathbb{P}^{U_1}(u) \\
&= \lim_{\eta \to 0} \frac{\mathbb{P}\left(\|\phi+\phi'\| \le \sqrt{\eta}\right)}{\mu^{d-1}(C_u(\eta))}. \qquad (3.6)
\end{aligned}$$

Since

$$\int_{\mathbb{S}^{d-1}} g(-u)\, d\mathbb{P}^{U_1}(u) = \int_{\mathbb{S}^{d-1}} g(u)g(-u)\, \mu^{d-1}(du),$$

(3.6) implies

$$\mathbb{P}\left(\|\phi+\phi'\| \le \sqrt{\eta}\right) \sim \mu^{d-1}(C_u(\eta)) \cdot \int_{\mathbb{S}^{d-1}} g(u)g(-u)\, \mu^{d-1}(du) \qquad (3.7)$$

as $\eta \to 0$.

Using the result derived in Appendix B.2 we obtain

$$\mu^{d-1}(C_u(\eta)) \sim \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot \eta^{\frac{d-1}{2}}$$

as $\eta \to 0$. Plugging this into (3.7), we have

$$\mathbb{P}(\|\phi\| \le \eta) \sim \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \eta^{\frac{d-1}{2}} \int_{\mathbb{S}^{d-1}} g(u)g(-u)\mu^{d-1}(du)$$

as $\eta \to 0$. ☐

Let $(y_1, y_2) \in [0,1]^2$ with $y_1 + y_2 \le \varepsilon$, $z = y_1 + y_2 \in [0, \varepsilon]$ and

$$\begin{aligned} \eta_1 &:= \eta_1(z) := 4(\varepsilon - z) - 7\varepsilon^2, \\ \eta_2 &:= \eta_2(z) := 4(\varepsilon - z) + 20\varepsilon^2. \end{aligned}$$

By Lemma 3.2 and Lemma 3.4 we have for $\varepsilon \to 0$

$$\begin{aligned} p_1(\varepsilon) &= \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \mathbb{P}(\|\phi\| \le \eta_1)\, d\mathbb{P}^Z(z) \\ &\sim \beta \cdot \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \left(4(\varepsilon - z) - 7\varepsilon^2\right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} p_2(\varepsilon) &= \int_{z=0}^{\varepsilon + 5\varepsilon^2} \mathbb{P}(\|\phi\| \le \eta_2)\, d\mathbb{P}^Z(z) \\ &\sim \beta \cdot \int_{z=0}^{\varepsilon + 5\varepsilon^2} \left(4(\varepsilon - z) + 20\varepsilon^2\right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z) \end{aligned} \tag{3.9}$$

with β given in (3.5).

Recall that $p_1(\varepsilon) \le p(\varepsilon) \le p_2(\varepsilon)$. If we can prove that $\lim_{\varepsilon \to 0} p_2(\varepsilon)/p_1(\varepsilon) = 1$, i.e.,

$$\int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \left(4(\varepsilon - z) - 7\varepsilon^2\right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z) \sim \int_{z=0}^{\varepsilon + 5\varepsilon^2} \left(4(\varepsilon - z) + 20\varepsilon^2\right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z)$$

as $\varepsilon \to 0$, we then obtain the asymptotic behavior of the tail probability $p(\varepsilon)$.

We consider again the distribution of the angle U_i. If

$$\int_{\mathbb{S}^{d-1}} g(u)g(-u)\mu^{d-1}(du) = 0,$$

then $g(u)g(-u) = 0$ μ^{d-1}-almost everywhere, which contradicts the assumption that the support of the point distribution is the unit ball. However, since g is bounded, we have

$$\int_{\mathbb{S}^{d-1}} g(u)g(-u)\mu^{d-1}(du) \in (0, \infty),$$

then β in (3.5) is a positive constant and by the dominated convergence theorem both the integrals in (3.8) and in (3.9) converge to zero as $\varepsilon \to 0$. Consequently, the tail probability converges to zero. Moreover, because of the boundedness, the density g of U_i also satisfies

$$\int_{\mathbb{S}^{d-1}} g(u)g(-u)^2 \mu^{d-1}(du) \in (0, \infty),$$

which is essential for proving (3.2).

Since a necessary condition for the distance exceeding a threshold $2 - \varepsilon$ is that the radii of the two points exceed $1 - \varepsilon$, it is not surprising that the tail of the distribution of R_i plays a central role. In the following sections we will investigate the limit distribution of the largest interpoint distance under specific conditions on the tail of the distribution of R_i.

3.3 Power type

Having proved the technical Lemma 3.4 and the expressions of the upper and the lower bound on the tail probability $p(\varepsilon)$, we now proceed to the main derivation of the extremal result under the assumption that the tail behavior of the distribution function of R_i can be approximated by a power function.

Denote by F the distribution function of $Y_i := 1 - R_i$, $i = 1, 2, \ldots$. Suppose that F is differentiable in a small right neighborhood of 0 and $F'(s) \sim a\alpha s^{\alpha-1}$ as $s \to 0$ for some $a > 0$ and $\alpha > 0$. By Lemma 3.3, we have $F(s) \sim as^{\alpha}$ as $s \to 0$. Since F is also continuous in a right neighborhood of 0, the distribution function of R_1 is $\mathbb{P}(R_1 \leq s) = 1 - F(1 - s)$ for sufficiently large $s \in [0, 1]$. The asymptotic expression for F implies that the distribution function of R_1 is tail-equivalent to the power distribution function $1 - (1 - t)^{\alpha} =: \Psi(t)$, i.e., we have

$$\lim_{t \to 1} \frac{\mathbb{P}(R_1 > t)}{1 - \Psi(t)} = \lim_{s \to 0} \frac{F(s)}{s^{\alpha}} = a > 0.$$

We therefore say that the distribution of radius is of power type.

The (right-hand) derivative of F at $s = 0$ is then given by

$$F'(0+) = \lim_{s \to 0} \frac{F(s) - F(0)}{s} = \lim_{s \to 0} \frac{F(s)}{s^{\alpha}} \cdot s^{\alpha-1} = \begin{cases} \infty, & \text{if } \alpha < 1 \\ a, & \text{if } \alpha = 1 \\ 0, & \text{if } \alpha > 1. \end{cases}$$

In other words, the probability distribution of R_1 becomes sparser when approaching its right endpoint 1 if $\alpha > 1$, it is homogeneous in the case $\alpha = 1$ and it becomes denser if $\alpha < 1$.

We first derive the tail behavior of the distribution of the distance between two points by investigating the asymptotic behavior of the upper bound $p_1(\varepsilon)$ and the lower bound $p_2(\varepsilon)$.

Proposition 3.5. *Let X_1, X_2 be i.i.d. points in $\mathbb{B}^d$, $d \geq 2$, such that $X_i = R_i U_i$, $i = 1, 2$, where $R_i = \|X_i\|$ and $U_i = X_i/\|X_i\| \in \mathbb{S}^{d-1}$ are independent. Suppose that U_1 has a bounded density g with respect to μ^{d-1} and that the distribution function F of $1 - R_1$ is differentiable in a small right neighborhood of 0 and satisfies*

$$F'(s) \sim a\alpha s^{\alpha-1} \tag{3.10}$$

as $s \to 0$ for some $a > 0$ and $\alpha > 0$. As $\varepsilon \to 0$, we then have

$$\mathbb{P}(\|X_1 - X_2\| \geq 2 - \varepsilon) \sim \sigma_1 \cdot \varepsilon^{\frac{d-1}{2} + 2\alpha},$$

where

$$\sigma_1 = \frac{(4\pi)^{\frac{d-1}{2}} a^2 \Gamma(\alpha + 1)^2}{\Gamma\left(\frac{d+1}{2} + 2\alpha\right)} \int_{\mathbb{S}^{d-1}} g(u) g(-u) \mu^{d-1}(du). \tag{3.11}$$

Proof. Let $Y_i := 1 - R_i$, $i = 1, 2$, and $Z := Y_1 + Y_2$. In the last section we proved that the tail probability $\mathbb{P}(\|X_1 - X_2\| \geq 2 - \varepsilon)$ satisfies

$$p_1(\varepsilon) \leq \mathbb{P}(\|X_1 - X_2\| \geq 2 - \varepsilon) \leq p_2(\varepsilon),$$

where

$$p_1(\varepsilon) \sim \beta \cdot \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \left(4(\varepsilon - z) - 7\varepsilon^2\right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z),$$

$$p_2(\varepsilon) \sim \beta \cdot \int_{z=0}^{\varepsilon + 5\varepsilon^2} \left(4(\varepsilon - z) + 20\varepsilon^2\right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z)$$

as $\varepsilon \to 0$. Note that β is a positive finite constant given in (3.5).

At first, we derive the asymptotic behavior of the distribution function of Z. By (3.10) and Lemma 3.3, Y_1 has the distribution function F with

$$F(s) \sim as^{\alpha} \tag{3.12}$$

as $s \to 0$. Since Y_1 and Y_2 are i.i.d., applying Lemma 3.3 and substituting $u = t/s$ lead to the following asymptotic expression of the distribution function of the convolution of Y_1 and Y_2:

$$\begin{aligned} F * F(s) &\sim \int_0^s a(s-t)^{\alpha} \cdot a\alpha t^{\alpha-1} dt \\ &= a^2 \alpha s^{2\alpha} \int_0^1 (1-u)^{\alpha} u^{\alpha-1} du \\ &= a^2 \alpha s^{2\alpha} \mathrm{B}(\alpha+1, \alpha) \\ &= \frac{a^2 \Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} s^{2\alpha} \end{aligned} \tag{3.13}$$

as $s \to 0$.

The main part of the proof is to compute the integrals in $p_1(\varepsilon)$ and $p_2(\varepsilon)$ as follows. Using integration by parts, we obtain

$$\begin{aligned} p_1(\varepsilon) &\sim \beta \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \left(4(\varepsilon - z) - 7\varepsilon^2 \right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z) \\ &= \beta \cdot 4^{\frac{d-1}{2}} \left[\left((\varepsilon - z) - \frac{7}{4}\varepsilon^2 \right)^{\frac{d-1}{2}} \mathbb{P}(Z \le z) \Big|_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \right. \\ &\qquad \left. + \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \frac{d-1}{2} \left((\varepsilon - z) - \frac{7}{4}\varepsilon^2 \right)^{\frac{d-3}{2}} \mathbb{P}(Z \le z)\, dz \right] \\ &= \beta \cdot 4^{\frac{d-1}{2}} \frac{d-1}{2} \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \left(\varepsilon - \frac{7}{4}\varepsilon^2 - z \right)^{\frac{d-3}{2}} F * F(z)\, dz. \end{aligned}$$

Applying the approximation (3.13) to the convolution $F * F$, Lemma 3.2 yields

$$p_1(\varepsilon) \sim \beta \cdot 4^{\frac{d-1}{2}} \frac{d-1}{2} \cdot \frac{a^2 \Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \left(\varepsilon - \frac{7}{4}\varepsilon^2 - z \right)^{\frac{d-3}{2}} z^{2\alpha}\, dz$$

as ε 0. By substituting $t = z/(\varepsilon - \frac{7}{4}\varepsilon^2)$, we have for ε 0

$$
\begin{aligned}
& p_1(\varepsilon) \\
\sim\ & \beta \cdot \frac{d-1}{2} \cdot \frac{4^{\frac{d-1}{2}} a^2 \Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} \cdot \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}+2\alpha} \int_{t=0}^{1} (1-t)^{\frac{d-3}{2}} t^{2\alpha}\, dt \\
=\ & \beta \cdot \frac{d-1}{2} \cdot \frac{4^{\frac{d-1}{2}} a^2 \Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} \cdot \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}+2\alpha} \mathrm{B}\left(\frac{d-1}{2}, 2\alpha+1\right) \\
=\ & \beta \cdot \frac{4^{\frac{d-1}{2}} a^2 \Gamma(\alpha+1)^2\, \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}+2\alpha\right)} \cdot \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}+2\alpha}.
\end{aligned}
$$

Plugging formula (3.5) for β into $p_1(\varepsilon)$, it reduces to

$$
p_1(\varepsilon) \ \sim\ \sigma_1 \cdot \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}+2\alpha}
$$

as ε 0, where σ_1 is the positive nite constant given in (3.11).

Using the same method and substituting $t = z/(\varepsilon + 5\varepsilon^2)$, we get

$$
\begin{aligned}
& p_2(\varepsilon) \\
\sim\ & \beta \int_{z=0}^{\varepsilon+5\varepsilon^2} \left(4(\varepsilon - z) + 20\varepsilon^2\right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z) \\
=\ & \beta \cdot 4^{\frac{d-1}{2}} \frac{d-1}{2} \int_{z=0}^{\varepsilon-5\varepsilon^2} \left(\varepsilon + 5\varepsilon^2 - z\right)^{\frac{d-3}{2}} F * F(z)\, dt \\
\sim\ & \beta \cdot \frac{d-1}{2} \cdot \frac{4^{\frac{d-1}{2}} a^2 \Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} \int_{z=0}^{\varepsilon+5\varepsilon^2} \left(\varepsilon + 5\varepsilon^2 - z\right)^{\frac{d-3}{2}} z^{2\alpha}\, dz \\
=\ & \beta \cdot \frac{d-1}{2} \cdot \frac{4^{\frac{d-1}{2}} a^2 \Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} \cdot \left(\varepsilon + 5\varepsilon^2\right)^{\frac{d-1}{2}+2\alpha} \int_{t=0}^{1} (1-t)^{\frac{d-3}{2}} t^{2\alpha}\, dt \\
=\ & \beta \cdot \frac{d-1}{2} \cdot \frac{4^{\frac{d-1}{2}} a^2 \Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} \cdot \left(\varepsilon + 5\varepsilon^2\right)^{\frac{d-1}{2}+2\alpha} \mathrm{B}\left(\frac{d-1}{2}, 2\alpha+1\right) \\
=\ & \sigma_1 \cdot \left(\varepsilon + 5\varepsilon^2\right)^{\frac{d-1}{2}+2\alpha}
\end{aligned}
$$

as ε 0.

Since $p_1(\varepsilon) \sim p_2(\varepsilon)$ as ε 0 and $p_1(\varepsilon) \le \sigma_1 \varepsilon^{\frac{d-1}{2}+2\alpha} \le p_2(\varepsilon)$, the result in Proposition 3.5 follows immediately. □

Thus, under the condition (3.10), the tail probability of the distance between X_1 and X_2 is asymptotically equal to some power function of ε. Indeed, we need (3.10) to be true to obtain (3.13). By Lemma 3.3, the "direct" condition (3.12) follows immediately from (3.10), and it coincides with the condition on F given in Theorem 3.1 in Lao and Mayer [27]. However, the converse does not always hold, e.g. for a Cantor-Lebesgue function.

A more general case of power type distribution function is regularly varying distribution function. Suppose that $F \in \mathcal{R}_\alpha$, $\alpha > 0$, where $\mathcal{R}_\alpha$ denotes the set of all functions which are regularly varying at 0 of index α (see Appendix C). Then, we have $F(s) \sim s^\alpha L(s)$ as $s \to 0$, where $L(s)$ is slowly varying at 0 (in case of power type distribution, $L(s) = a$). By the monotone density theorem stated in Appendix C, we have $F'(s) \sim \alpha s^{\alpha-1} L(s)$ as $s \to 0$. Thus, using the substitution $u = t/s$, the convolution of F has the asymptotic expression

$$
\begin{aligned}
F * F(s) &\sim \int_0^s (s-t)^\alpha L(s-t) \cdot \alpha t^{\alpha-1} L(t)\, dt \\
&= \alpha L(s)^2 s^{2\alpha} \int_0^1 (1-u)^\alpha \frac{L(s(1-u))}{L(s)} u^{\alpha-1} \frac{L(su)}{L(s)}\, du \\
&\sim \alpha L(s)^2 s^{2\alpha} \mathrm{B}(\alpha+1, \alpha) = \frac{\Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} L(s)^2 s^{2\alpha}
\end{aligned}
$$

as $s \to 0$, where we use the property $\lim_{s\to 0} L(st)/L(s) = 1$, $t > 0$, for slowly varying functions. The asymptotic expression of the tail probability can be obtained similarly, as $\varepsilon \to 0$ we have

$$
\mathbb{P}(\|X_1 + X_2\| \geq 2 - \varepsilon) \sim \beta\, 4^{\frac{d-1}{2}} \frac{\Gamma(\alpha+1)^2\, \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+1}{2} + 2\alpha\right)} \varepsilon^{\frac{d-1}{2}+2\alpha} L(\varepsilon)^2.
$$

To deduce the limit law of D_n, we still need more information about $L(\varepsilon)$.

We return to the situation of power type distributions. The main result of this section is as follows:

Theorem 3.6. *Let $X_1, X_2, \ldots$ be i.i.d. points in $\mathbb{B}^d$, $d \geq 2$, such that $X_i = R_i U_i$, $i = 1, 2, \ldots$, where $R_i = \|X_i\|$ and $U_i = X_i/\|X_i\| \in \mathbb{S}^{d-1}$ are independent. Suppose that U_1 has a bounded density g with respect to μ^{d-1} and that condition (3.10) holds for the distribution function F of $1 - R_1$. We then have*

$$
\lim_{n\to\infty} \mathbb{P}\left(\left(\frac{\sigma_1}{2}\right)^{\frac{2}{d-1+4\alpha}} \cdot n^{\frac{4}{d-1+4\alpha}} \cdot (2 - D_n) \leq t\right) = 1 - \exp\left(-t^{\frac{d-1}{2}+2\alpha}\right)
$$

for $t > 0$, where σ_1 is given in (3.11).

Proof. First, we set

$$\varepsilon_n = \varepsilon_n(t) := \left(\frac{2}{\sigma_1}\right)^{\frac{2}{d-1+4\alpha}} \cdot n^{-\frac{4}{d-1+4\alpha}} \cdot t \tag{3.14}$$

for fixed $t > 0$. We check the conditions (3.1) and (3.2) of Lemma 3.1. If they are satisfied, then

$$\begin{aligned}
\mathbb{P}(2 - D_n \le \varepsilon_n) &= \mathbb{P}\left(2 - D_n \le \left(\frac{2}{\sigma_1}\right)^{\frac{2}{d-1+4\alpha}} \cdot n^{-\frac{4}{d-1+4\alpha}} \cdot t\right) \\
&= \mathbb{P}\left(\left(\frac{\sigma_1}{2}\right)^{\frac{2}{d-1+4\alpha}} \cdot n^{\frac{4}{d-1+4\alpha}} \cdot (2 - D_n) \le t\right) \\
&\to 1 - \exp(-\lambda(t))
\end{aligned}$$

as $n \to \infty$, where $\lambda(t)$ will be proved to be a positive finite constant as follows.

Applying Proposition 3.5 and plugging (3.14) into the tail probability, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \binom{n}{2} \mathbb{P}(\|X_1 - X_2\| > 2 - \varepsilon_n) \\
&= \lim_{n\to\infty} \frac{n^2}{2} \cdot \sigma_1 \cdot \varepsilon_n^{\frac{d-1}{2}+2\alpha} \\
&= \lim_{n\to\infty} \frac{\sigma_1}{2} \cdot n^2 \cdot \frac{2}{\sigma_1} \cdot n^{-2} \cdot t^{\frac{d-1}{2}+2\alpha} \\
&= t^{\frac{d-1}{2}+2\alpha} =: \lambda(t) \in (0, \infty).
\end{aligned}$$

Consequently, condition (3.1) holds.

To check condition (3.2) we will use the independence of the i.i.d. points to obtain an upper bound for the probability of two overlapping exceedances. Write $X_i = U_i R_i$, $i = 1, 2, 3$, and put $Y_i = 1 - R_i$, $i = 1, 2, 3$. Then Y_1, Y_2, Y_3 are i.i.d. with the distribution function F, where

$$F(s) \sim a s^{\alpha}$$

as $s \to 0$ by Lemma 3.3. Put $\phi = \measuredangle(-U_1, U_2)$ and $\phi' = \measuredangle(-U_1, U_3)$. By adopting the same approach as in Section 3.2 and using the independence of Y_i and U_i,

$i = 1, 2, 3$, we obtain

$$
\begin{aligned}
& n^3 \mathbb{P}\big(\|X_1 - X_2\| > 2 - \varepsilon_n,\ \|X_1 - X_3\| > 2 - \varepsilon_n \big) \\
= \ & n^3 \mathbb{P}\Big(\phi^2 \cos \xi \le 4(\varepsilon_n - Y_1 - Y_2) + \varrho(\varepsilon_n, Y_1, Y_2), \\
& \qquad \phi'^2 \cos \xi' \le 4(\varepsilon_n - Y_1 - Y_3) + \varrho(\varepsilon_n, Y_1, Y_3), \\
& \qquad\qquad Y_1 + Y_2 \le \varepsilon_n, Y_1 + Y_3 \le \varepsilon_n \Big) \\
\le \ & n^3 \mathbb{P}\Big(|\phi| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}},\ |\phi'| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}},\ Y_i \le \varepsilon_n, i = 1, 2, 3 \Big) \\
= \ & n^3 F(\varepsilon_n)^3 \mathbb{P}\Big(|\phi| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}},\ |\phi'| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big),
\end{aligned}
\tag{3.15}
$$

where $\varrho(\varepsilon_n, Y_1, Y_2)$ and $\varrho(\varepsilon_n, Y_1, Y_3)$ are given in (3.4) and $\xi \in [0, \phi]$, $\xi' \in [0, \phi']$.

We now deal with the probability in the last expression. To this end, let $\phi_u = \measuredangle(-u, U_2)$ and $\phi'_u = \measuredangle(-u, U_3)$, $u \in \mathbb{S}^{d-1}$. Since ϕ_u and ϕ'_u are i.i.d., it follows that

$$
\begin{aligned}
& \mathbb{P}\Big(|\phi| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}},\ |\phi'| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big) \\
= \ & \int_{\mathbb{S}^{d-1}} \mathbb{P}\Big(|\phi_u| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}},\ |\phi'_u| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big) g(u)\, \mu^{d-1}(du) \\
= \ & \int_{\mathbb{S}^{d-1}} \mathbb{P}\Big(|\phi_u| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big)^2 g(u)\, \mu^{d-1}(du).
\end{aligned}
$$

As in the proof of Lemma 3.4 we have for each fixed $u \in \mathbb{S}^{d-1}$

$$
\begin{aligned}
\mathbb{P}\Big(|\phi_u| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big) & \sim g(-u) \cdot \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot (4\varepsilon_n + 20\varepsilon_n^2)^{\frac{d-1}{2}} \\
& \sim g(-u) \cdot \frac{(4\pi)^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot \varepsilon_n^{\frac{d-1}{2}}
\end{aligned}
$$

as $\varepsilon_n \to 0$. Since g is a bounded function, the dominated convergence theorem yields

$$
\begin{aligned}
& \mathbb{P}\Big(|\phi| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}},\ |\phi'| \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big) \\
\sim \ & \frac{(4\pi)^{d-1}}{\Gamma\left(\frac{d+1}{2}\right)^2} \cdot \varepsilon_n^{d-1} \cdot \int_{\mathbb{S}^{d-1}} g(-u)^2 g(u)\, \mu^{d-1}(du) \ =: \ C \cdot \varepsilon_n^{d-1},
\end{aligned}
\tag{3.16}
$$

where C is some finite positive constant because of the boundedness of g.

Thus, applying the approximations $F(\varepsilon_n) \sim a\varepsilon_n^{\alpha}$ and (3.16) as $\varepsilon_n \to 0$ and plugging (3.14) into the upper bound (3.15), we get

$$\begin{aligned}
& n^3 F(\varepsilon_n)^3 \mathbb{P}\Big(\phi \le (4\varepsilon_n + 20\varepsilon_n^2)^{\frac{1}{2}}, \quad \phi \le (4\varepsilon_n + 20\varepsilon_n^2)^{\frac{1}{2}} \Big) \\
\sim \quad & n^3 \cdot C \cdot \varepsilon_n^{d-1} \cdot a^3 \varepsilon_n^{3\alpha} \\
= \quad & C \cdot a^3 \cdot \left(\frac{2}{\sigma_1}\right)^{\frac{2(d-1+3\alpha)}{d-1+4\alpha}} \cdot t^{d-1+3\alpha} \cdot n^{-\frac{d-1}{d-1+4\alpha}} \\
\to \quad & 0,
\end{aligned}$$

where asymptotic equality and convergence refer to $n \to \infty$. Hence, the main result follows directly from Lemma 3.1. □

So far, we did not impose any restriction on the distribution of the angle except for the boundedness of g. A large class of multivariate distributions is the class of spherically symmetric distributions, for which the random points have independent radii R_i and directions U_i. Moreover, the directions are uniformly distributed on $\mathbb{S}^{d-1}$, e.g. U_i has the density

$$g_0(u) = \frac{1}{\mu^{d-1}(\mathbb{S}^{d-1})}\mathbf{1}_{\mathbb{S}^{d-1}}(u) = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}}\mathbf{1}_{\mathbb{S}^{d-1}}(u), \qquad u \in \mathbb{R}^d, \tag{3.17}$$

with respect to μ^{d-1}. In this case the integral figuring in (3.11) takes the value

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} g_0(u) g_0(-u) \mu^{d-1}(du) &= \int_{\mathbb{S}^{d-1}} g_0(u)^2 \, \mu^{d-1}(du) \\
&= \int_{\mathbb{S}^{d-1}} \frac{1}{\mu^{d-1}(\mathbb{S}^{d-1})^2} \, \mu^{d-1}(du) \\
&= \frac{1}{\mu^{d-1}(\mathbb{S}^{d-1})} = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}}.
\end{aligned}$$

Plugging this result into the assertion of Theorem 3.6, we get the following corollary, which has also been proved in [32] using a different method.

Corollary 3.7. *If the i.i.d. points $X_1, X_2, \ldots$ have a spherically symmetric distribution in $\mathbb{B}^d$, $d \ge 2$, and condition (3.12) holds for the distribution function of* $1 - \|X_1\|$, *we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\left(\frac{\sigma_1}{2}\right)^{\frac{2}{d-1+4\alpha}} \cdot n^{\frac{4}{d-1+4\alpha}} \cdot (2 - D_n) \le t\right) = 1 - \exp\left(- t^{\frac{d-1}{2}+2\alpha}\right),$$

$t > 0$, *where*

$$\sigma_1 = \frac{2^{d-1}\Gamma\left(\frac{d}{2}\right) a^2 \Gamma(\alpha+1)^2}{\sqrt{\pi}\,\Gamma\left(\frac{d+1}{2}+2\alpha\right)}.$$

For the special case of uniformly distributed points in $\mathbb{B}^d$, i.e. the case $\alpha = 1$ and $a = d$, the limit distribution is as follows (cf. [26] and [32]).

Corollary 3.8. *If $X_1, X_2, \ldots$ are independent and uniformly distributed in $\mathbb{B}^d$, $d \geq 2$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\left(\frac{\sigma_1}{2}\right)^{\frac{2}{d+3}} \cdot n^{\frac{4}{d+3}} \cdot (2 - D_n) \leq t\right) = 1 - \exp\left(-t^{\frac{d+3}{2}}\right), \tag{3.18}$$

$t > 0$, *where*

$$\sigma_1 = \frac{2^{d+1} d}{(d+1)(d+3)\mathrm{B}\left(\frac{d}{2}+\frac{1}{2}, \frac{1}{2}\right)}.$$

Figure 3.2 shows a simulation of the limit law of the largest interpoint distance between points in the unit circle. The solid curves are the empirical distribution function (EDF) of $(\frac{16}{15\pi})^{2/5} \cdot n^{4/5} \cdot (2 - D_n)$ with $n = 1000$ (lower curve) and $n = 100000$ (upper curve), respectively. The dotted smooth curve is the theoretical distribution function on the right-hand side of (3.18). We see that the convergence of the EDF to the limit law is slow, so that we need a relatively large sample size to get a good approximation.

If the parameters α, a and d are fixed, the limit distribution in Theorem 3.6 depends only on the value of the integral $\int_{\mathbb{S}^{d-1}} g(u)g(-u)\mu^{d-1}(du)$. Denote by g_0 the uniform density on $\mathbb{S}^{d-1}$ given in (3.17) and by $\mathcal{G}_c$ the class of bounded centrally symmetric densities with respect to μ^{d-1}, i.e., we have $g(u) = g(-u)$, $u \in \mathbb{S}^{d-1}$, for each $g \in \mathcal{G}_c$. By the Cauchy-Schwarz inequality, we get

$$\left(\int_{\mathbb{S}^{d-1}} g(u)g_0(u)\,\mu^{d-1}(du)\right)^2 \leq \int_{\mathbb{S}^{d-1}} g(u)^2\,\mu^{d-1}(du) \cdot \int_{\mathbb{S}^{d-1}} g_0(u)^2\,\mu^{d-1}(du).$$

Since

$$\int_{\mathbb{S}^{d-1}} g(u)g_0(u)\,\mu^{d-1}(du) = 1/\mu^{d-1}(\mathbb{S}^{d-1}) = \int_{\mathbb{S}^{d-1}} g_0(u)^2\,\mu^{d-1}(du),$$

it follows that

$$\int_{\mathbb{S}^{d-1}} g(u)^2\,\mu^{d-1}(du) \geq \frac{1}{\mu^{d-1}(\mathbb{S}^{d-1})} = \int_{\mathbb{S}^{d-1}} g_0(u)^2\,\mu^{d-1}(du),$$

i.e., the integral $\int_{\mathbb{S}^{d-1}} g(u)g(-u)\mu^{d-1}(du)$ takes its minimum for $g = g_0$, and we have the following corollary:

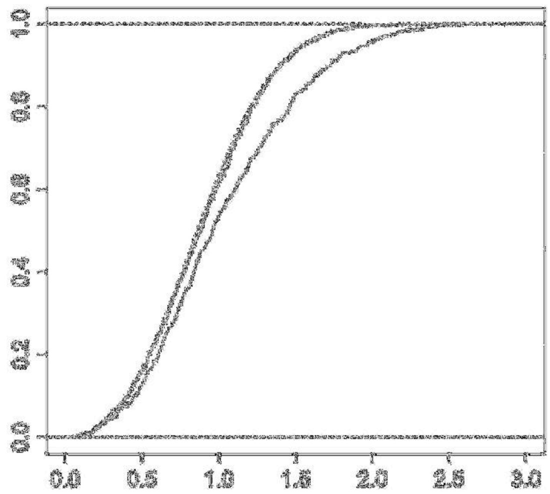

Figure 3.2: EDFs of $(\frac{16}{15\pi})^{2/5} \cdot n^{4/5} \cdot (2 - D_n)$ with $n = 1000$ (lower curve) and $n = 10^5$ (upper curve). The dotted smooth curve is the limit law $1 - \exp\left(-t^{5/2}\right)$.

Corollary 3.9. *Among the class $\mathcal{G}_c$ of centrally symmetric densities, the U-max-statistic D_n is asymptotically stochastically minimal if the direction has a uniform distribution on the surface of the unit ball.*

Up to now, we have discussed the limit law of the largest interpoint distance D_n, when the distribution of X_1 belongs to some class of power type distributions. In this case the limit law of D_n is of Weibull type, and the rescaling factor is of the order $O\left(n^{\frac{4}{d-1+4\alpha}}\right)$.

3.4 Logarithmic type

In this section we investigate the limit distribution of the largest interpoint distance for the case that the distribution function of R_1 behaves like a logarithmic function near 1.

Suppose that the distribution function F of $Y_1 = 1 - R_1$ is differentiable in a small right neighborhood of 0, and that

$$F'(s) \sim \frac{a\alpha}{(1 - \alpha \log s)^2} \cdot \frac{1}{s}$$

as $s \to 0$ for some $a > 0$ and $\alpha > 0$. Thus, by Lemma 3.3 F has the asymptotic expression

$$F(s) \sim \int_0^s \frac{a\alpha}{(1-\alpha\log t)^2}\cdot\frac{1}{t}\,dt = \frac{a}{1-\alpha\log s}$$

as $s \to 0$. Since F is right-hand continuous at 0, then $1 - F(1-s)$ is the distribution function of R_1 at least for sufficiently large $s \in [0,1]$. The asymptotic expression above then implies the tail-equivalence of the distribution of R_1 and the logarithmic distribution function $1 - \frac{1}{1-\alpha\log(1-t)} =: \Psi_\alpha(t)$, i.e., we have

$$\lim_{t\to 1}\frac{\mathbb{P}(R_1 > t)}{1-\Psi_\alpha(t)} = \lim_{s\to 0} F(s)\cdot(1-\alpha\log s) = a > 0.$$

In this case, we say that the distribution of $\|X_1\|$ is of logarithmic type.

Notice that the distribution function F of logarithmic type belongs to the class of slowly varying functions, because of

$$\lim_{s\to 0}\frac{F(st)}{F(s)} = \lim_{s\to 0}\frac{1-\alpha\log s}{1-\alpha\log s-\alpha\log t} = 1.$$

We now consider the (right-hand) derivative of F at 0. Using l'Hôpital's rule we get

$$\begin{aligned} F'(0+) &= \lim_{s\to 0}\frac{F(s)-F(0)}{s} \\ &= \lim_{s\to 0}\frac{F(s)}{a/(1-\alpha\log s)}\cdot\lim_{s\to 0}\frac{as^{-1}}{1-\alpha\log s} \\ &= 1\cdot\frac{a}{\alpha}\lim_{s\to 0} s^{-1} \\ &= \infty. \end{aligned}$$

Actually, the speed of convergence of the derivative of F at 0 to infinity is even more rapid than that of a distribution function of power type with $\alpha < 1$ (see page 23), a phenomenon that is illustrated in Figure 3.3.

Since the random points are more likely to fall in a narrow annulus close to the boundary than in the case of a radial distribution function of power type, the probability of an exceedance over a given threshold is greater. To compensate for this effect with respect to the situation of Theorem 3.6, we need a rescaling factor which converges to infinity more rapidly.

The following proposition gives the tail behavior of the distribution of the distance between two points, in the setting of Section 3.4.

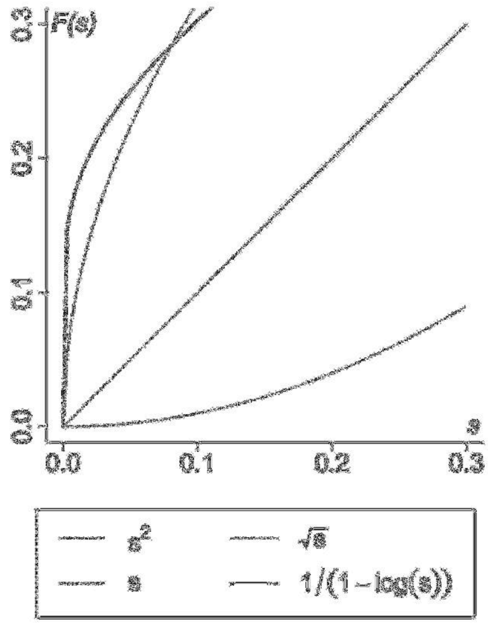

Figure 3.3: The function $1/(1 - \log s)$ decays more rapidly than all power type distribution functions as $s \to 0$.

Proposition 3.10. *Let X_1, X_2 be i.i.d. points in $\mathbb{B}^d$, $d \geq 2$, such that $X_i = R_i U_i$, $i = 1, 2$, where $R_i = \|X_i\|$ and $U_i = X_i/\|X_i\| \in \mathbb{S}^{d-1}$ are independent. Suppose that U_1 has a bounded density g with respect to μ^{d-1} and that the distribution function F of $1 - R_1$ is differentiable in a small neighborhood of 0 and satisfies*

$$F'(s) \sim \frac{a\alpha}{(1 - \alpha \log s)^2} \cdot \frac{1}{s} \tag{3.19}$$

as $s \to 0$ for some $a > 0$ and $\alpha > 0$. As $\varepsilon \to 0$, we then have

$$\mathbb{P}(\|X_1 - X_2\| \geq 2 - \varepsilon) \sim \sigma_2 \cdot (\log \varepsilon)^{-2} \cdot \varepsilon^{\frac{d-1}{2}}$$

as $\varepsilon \to 0$, where

$$\sigma_2 = \frac{(4\pi)^{\frac{d-1}{2}} a^2}{\alpha^2 \Gamma\left(\frac{d+1}{2}\right)} \int_{\mathbb{S}^{d-1}} g(u) g(-u) \mu^{d-1}(du). \tag{3.20}$$

Proof. As in Proposition 3.5, we deduce the asymptotic behavior of the bounds on the tail probability.

Let $Y_i = 1 - R_i$, $i = 1, 2$. The i.i.d. random variables Y_1 and Y_2 have the distribution function F, where

$$F(s) \sim \frac{a}{1 - \alpha \log s}$$

as $s \to 0$, which follows from (3.19) and Lemma 3.3. As before, we put $Z = Y_1 + Y_2$. We first derive the asymptotic behavior of the convolution of F with itself. Lemma 3.3 yields

$$F * F(s) \sim \int_0^s \frac{a}{1-\alpha\log(s-t)} \cdot \frac{a\alpha}{(1-\alpha\log t)^2} \cdot \frac{1}{t}\, dt.$$

In the following, we derive an upper and a lower bound for the right-hand side. Since the function $1/(1-\alpha\log(s-t))$ is decreasing in t on the interval $[0, s]$, we have

$$\begin{aligned}
& a^2\alpha \int_0^s \frac{1}{1-\alpha\log(s-t)} \cdot \frac{1}{(1-\alpha\log t)^2} \cdot \frac{1}{t}\, dt \\
\leq\ & \frac{a^2\alpha}{1-\alpha\log s} \cdot \int_0^s \frac{1}{(1-\alpha\log t)^2} \cdot \frac{1}{t}\, dt \\
=\ & \frac{a^2\alpha}{1-\alpha\log s} \cdot \frac{1}{\alpha} \cdot \frac{1}{1-\alpha\log t}\Bigg|_{t=0}^{s} \\
=\ & \frac{a^2}{(1-\alpha\log s)^2} \\
\sim\ & \frac{a^2}{\alpha^2(\log s)^2},
\end{aligned}$$

where "$\sim$" refers to $s \to 0$. To find a lower bound, we confine the integration to the interval $[0, s/2]$ and then use the inequality $\frac{1}{1-\alpha\log(s-t)} \geq \frac{1}{1-\alpha\log(s/2)}$ for $t \in [0, s/2]$. We thus obtain

$$\begin{aligned}
& a^2\alpha \int_0^s \frac{1}{1-\alpha\log(s-t)} \cdot \frac{1}{(1-\alpha\log t)^2} \cdot \frac{1}{t}\, dt \\
\geq\ & \frac{a^2\alpha}{1-\alpha\log(s/2)} \cdot \int_0^{s/2} \frac{1}{(1-\alpha\log t)^2} \cdot \frac{1}{t}\, dt \\
=\ & \frac{a^2\alpha}{1-\alpha\log(s/2)} \cdot \frac{1}{\alpha} \cdot \frac{1}{1-\alpha\log t}\Bigg|_{t=0}^{s/2} \\
=\ & \frac{a^2}{(1-\alpha\log(s/2))^2} \\
\sim\ & \frac{a^2}{\alpha^2(\log s)^2},
\end{aligned}$$

where "$\sim$" refers to $s \to 0$. Hence, we conclude that

$$F * F(s) \sim \frac{a^2}{\alpha^2(\log s)^2} \tag{3.21}$$

as $s \to 0$.

Now, we deal with the lower bound $p_1(\varepsilon)$ given in (3.8). Using integration by parts, we have

$$\begin{aligned} p_1(\varepsilon) &\sim \beta \cdot \int_{z=0}^{\varepsilon-\frac{7}{4}\varepsilon^2} \left(4(\varepsilon - z) - 7\varepsilon^2\right)^{\frac{d-1}{2}} d\mathbb{P}^Z(z) \\ &= \beta \cdot 4^{\frac{d-1}{2}} \left((\varepsilon - z) - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}} \mathbb{P}(Z \le z)\bigg|_{z=0}^{\varepsilon-\frac{7}{4}\varepsilon^2} \\ &\quad + \int_{z=0}^{\varepsilon-\frac{7}{4}\varepsilon^2} \frac{d-1}{2}\left((\varepsilon - z) - \frac{7}{4}\varepsilon^2\right)^{\frac{d-3}{2}} \mathbb{P}(Z \le z)\, dz \\ &= \beta \cdot 4^{\frac{d-1}{2}} \int_{z=0}^{\varepsilon-\frac{7}{4}\varepsilon^2} \frac{d-1}{2}\left(\varepsilon - \frac{7}{4}\varepsilon^2 - z\right)^{\frac{d-3}{2}} F * F(z)\, dz. \end{aligned}$$

Applying (3.21) on the convolution $F * F$ and using Lemma 3.2, the substitution $t = z/(\varepsilon - \frac{7}{4}\varepsilon^2)$ yields

$$\begin{aligned} &p_1(\varepsilon) \\ &\sim \beta \cdot 4^{\frac{d-1}{2}} \cdot \frac{a^2}{\alpha^2} \int_{z=0}^{\varepsilon-\frac{7}{4}\varepsilon^2} \frac{d-1}{2}\left(\varepsilon - \frac{7}{4}\varepsilon^2 - z\right)^{\frac{d-3}{2}} \frac{1}{(\log z)^2}\, dz \\ &= \beta \cdot 4^{\frac{d-1}{2}} \cdot \frac{a^2}{\alpha^2}\left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}} \int_{t=0}^{1} \frac{d-1}{2}(1-t)^{\frac{d-3}{2}} \frac{1}{(\log((\varepsilon - \frac{7}{4}\varepsilon^2)t))^2}\, dt \\ &= \frac{\beta \cdot 4^{\frac{d-1}{2}} a^2 \cdot \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}}}{\alpha^2(\log(\varepsilon - \frac{7}{4}\varepsilon^2))^2} \int_{t=0}^{1} \frac{d-1}{2}(1-t)^{\frac{d-3}{2}} \frac{(\log(\varepsilon - \frac{7}{4}\varepsilon^2))^2}{(\log((\varepsilon - \frac{7}{4}\varepsilon^2)t))^2}\, dt. \end{aligned}$$

Since for each fixed $t \in (0, 1)$

$$\lim_{\varepsilon \to 0} \frac{(\log(\varepsilon - \frac{7}{4}\varepsilon^2))^2}{(\log((\varepsilon - \frac{7}{4}\varepsilon^2)t))^2} = 1 \qquad \text{and} \qquad \left|\frac{(\log(\varepsilon - \frac{7}{4}\varepsilon^2))^2}{(\log((\varepsilon - \frac{7}{4}\varepsilon^2)t))^2}\right| \le 1,$$

by the dominated convergence theorem we have

$$
\begin{aligned}
p_1(\varepsilon) &\sim \frac{\beta \cdot 4^{\frac{d-1}{2}} a^2 \cdot \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}}}{\alpha^2 (\log(\varepsilon - \frac{7}{4}\varepsilon^2))^2} \int_{t=0}^{1} \frac{d-1}{2}(1-t)^{\frac{d-3}{2}}\, dt \\
&= \frac{\beta \cdot 4^{\frac{d-1}{2}} \cdot a^2}{\alpha^2 (\log(\varepsilon - \frac{7}{4}\varepsilon^2))^2} \cdot \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}} \\
&= \frac{\sigma_2}{(\log(\varepsilon - \frac{7}{4}\varepsilon^2))^2} \cdot \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}}
\end{aligned}
$$

as $\varepsilon \to 0$.

Similarly, applying integration by parts, substituting $t = z/(\varepsilon + 5\varepsilon^2)$ and using some asymptotic considerations, the upper bound $p_2(\varepsilon)$ is asymptotically equivalent to

$$
\begin{aligned}
p_2(\varepsilon) &\sim \beta \cdot 4^{\frac{d-1}{2}} \int_{z=0}^{\varepsilon+5\varepsilon^2} \frac{d-1}{2} \left(\varepsilon + 5\varepsilon^2 - z\right)^{\frac{d-3}{2}} F * F(z)\, dz \\
&\sim \frac{\beta \cdot 4^{\frac{d-1}{2}} a^2 \cdot \left(\varepsilon + 5\varepsilon^2\right)^{\frac{d-1}{2}}}{\alpha^2 (\log(\varepsilon + 5\varepsilon^2))^2} \int_{t=0}^{1} \frac{d-1}{2}(1-t)^{\frac{d-3}{2}}\, dt \\
&= \frac{\sigma_2}{(\log(\varepsilon + 5\varepsilon^2))^2} \cdot \left(\varepsilon + 5\varepsilon^2\right)^{\frac{d-1}{2}}
\end{aligned}
$$

as $\varepsilon \to 0$.

Since by l'Hôpital's rule we have

$$
\lim_{\varepsilon \to 0} \frac{\log(\varepsilon - \frac{7}{4}\varepsilon^2)}{\log(\varepsilon + 5\varepsilon^2)} = \lim_{\varepsilon \to 0} \frac{(1 - \frac{7}{2}\varepsilon)(\varepsilon + 5\varepsilon^2)}{(\varepsilon - \frac{7}{4}\varepsilon^2)(1 + 10\varepsilon)} = 1,
$$

it is convenient to verify that

$$
\lim_{\varepsilon \to 0} \frac{p_2(\varepsilon)}{p_1(\varepsilon)} = \lim_{\varepsilon \to 0} \left(\frac{\varepsilon + 5\varepsilon^2}{\varepsilon - \frac{7}{4}\varepsilon^2}\right)^{\frac{d-1}{2}} \left(\frac{\log(\varepsilon - \frac{7}{4}\varepsilon^2)}{\log(\varepsilon + 5\varepsilon^2)}\right)^2 = 1.
$$

Recall $p_1(\varepsilon) \le \mathbb{P}(\|X_1 - X_2\| \ge 2 - \varepsilon) \le p_2(\varepsilon)$. Combining this and

$$
\frac{\sigma_2 \left(\varepsilon - \frac{7}{4}\varepsilon^2\right)^{\frac{d-1}{2}}}{(\log(\varepsilon - \frac{7}{4}\varepsilon^2))^2} \le \frac{\sigma_2 \varepsilon^{\frac{d-1}{2}}}{(\log \varepsilon)^2} \le \frac{\sigma_2 \left(\varepsilon + 5\varepsilon^2\right)^{\frac{d-1}{2}}}{(\log(\varepsilon + 5\varepsilon^2))^2}
$$

for sufficiently small ε, the proof of the proposition is completed. □

We have obtained the tail probability of the distance over the threshold $2-\varepsilon$. Thus, Lemma 3.1 leads to the following theorem:

Theorem 3.11. *Let $X_1, X_2, \dots$ be i.i.d. points in $\mathbb{B}^d$, $d \geq 2$, such that $X_i = R_i U_i$, $i = 1, 2, \dots$, where $R_i = \|X_i\|$ and $U_i = X_i/\|X_i\| \in \mathbb{S}^{d-1}$ are independent. Suppose that U_1 has a bounded density g with respect to μ^{d-1} and that condition (3.19) holds for the distribution function F of $1 - R_1$. We then have*

$$\lim_{n\to\infty} \mathbb{P}\left(\left(\frac{(d-1)^2\sigma_2}{32}\right)^{\frac{2}{d-1}} \cdot \left(\frac{n}{\log n}\right)^{\frac{4}{d-1}} \cdot (2 - D_n) \leq t \right) = 1 - \exp\left(-t^{\frac{d-1}{2}}\right)$$

for $t > 0$, where σ_2 is given in (3.20).

Proof. For a fixed real number $t > 0$, put

$$\varepsilon_n = \varepsilon_n(t) := \left(\frac{32}{(d-1)^2\sigma_2}\right)^{\frac{2}{d-1}} \cdot \left(\frac{n}{\log n}\right)^{-\frac{4}{d-1}} \cdot t. \tag{3.22}$$

Write $l_2 n = \log\log n$ for short. Thus,

$$\log \varepsilon_n = \frac{2}{d-1}\log\left(\frac{32}{(d-1)^2\sigma_2}\right) - \frac{4}{d-1}\log n + \frac{4}{d-1} l_2 n + \log t.$$

If conditions (3.1) and (3.2) are satisfied, we use Lemma 3.1 to derive the limit law of D_n as follows (cf. Section 3.3):

$$\begin{aligned}
\mathbb{P}(2 - D_m \leq \varepsilon_n) &= \mathbb{P}\left(2 - D_n \leq \left(\frac{32}{(d-1)^2\sigma_2}\right)^{\frac{2}{d-1}} \cdot \left(\frac{n}{\log n}\right)^{-\frac{4}{d-1}} \cdot t\right) \\
&= \mathbb{P}\left(\left(\frac{(d-1)^2\sigma_2}{32}\right)^{\frac{2}{d-1}} \cdot \left(\frac{n}{\log n}\right)^{\frac{4}{d-1}} \cdot (2 - D_n) \leq t\right) \\
&\to 1 - \exp(\lambda(t))
\end{aligned}$$

as $n \to \infty$ with $\lambda(t) \in (0, \infty)$.

Putting ε_n and $\log \varepsilon_n$ into (3.1) and using Proposition 3.10, we have

$$\begin{aligned}
&\lim_{n\to\infty} \binom{n}{2} \mathbb{P}(\|X_1 - X_2\| > 2 - \varepsilon_n) \\
&= \lim_{n\to\infty} \frac{n^2}{2} \cdot \sigma_2 \cdot (\log \varepsilon_n)^{-2} \cdot \varepsilon_n^{\frac{d-1}{2}} \\
&= \lim_{n\to\infty} \left(\frac{1}{2\log n}\log\left(\frac{32}{(d-1)^2\sigma_2}\right) - 1 + \frac{l_2 n}{\log n} + \frac{(d-1)\log t}{4\log n}\right)^{-2} t^{\frac{d-1}{2}} \\
&= t^{\frac{d-1}{2}} =: \lambda(t).
\end{aligned}$$

Since $t \in (0, \infty)$, (3.1) is satisfied.

The proof of the second condition is also analogous to the proof of Theorem 3.6. Put $\phi_2 = \measuredangle(-U_1, U_2)$ and $\phi_3 = \measuredangle(-U_1, U_3)$. As in (3.15), we use the independence of R_i and U_i, $i = 1, 2, 3$, to obtain an upper bound on the probability of two overlapping exceedances:

$$
\begin{aligned}
& n^3 \mathbb{P}(\|X_1 - X_2\| > 2 - \varepsilon_n,\ \|X_1 - X_3\| > 2 - \varepsilon_n) \\
\le\ & n^3 F(\varepsilon_n)^3 \mathbb{P}\Big(\phi_2 \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}}, \quad \phi_3 \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big),
\end{aligned}
$$

where

$$
\begin{aligned}
& \mathbb{P}\Big(\phi_2 \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}}, \quad \phi_3 \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big) \\
\sim\ & \frac{(4\pi)^{d-1}}{\Gamma\left(\frac{d+1}{2}\right)^2} \cdot \varepsilon_n^{d-1} \cdot \int_{\mathbb{S}^{d-1}} g(-u)^3 g(u)\, \mu^{d-1}(du) \\
=:\ & C \cdot \varepsilon_n^{d-1}
\end{aligned}
$$

as $\varepsilon_n \to 0$ is given in (3.16) and

$$
F(\varepsilon_n) \sim \frac{a}{1 - \alpha \log \varepsilon_n}
$$

as $\varepsilon_n \to 0$ is the distribution function of $Y_i = 1 - R_i$. Plugging (3.22) into the upper bound, we have

$$
\begin{aligned}
& n^3 F(\varepsilon_n)^3 \mathbb{P}\Big(\phi_2 \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}}, \quad \phi_3 \le \big(4\varepsilon_n + 20\varepsilon_n^2\big)^{\frac{1}{2}} \Big) \\
\sim\ & n^3 \cdot C\varepsilon_n^{d-1} \cdot \left(\frac{a}{1 - \alpha \log \varepsilon_n} \right)^3 \\
=\ & n^3 \cdot C \cdot \left(\frac{32}{(d-1)^2 \sigma_2} \right)^2 \left(\frac{n}{\log n} \right)^{-4} t^{d-1} a^3 \\
& \cdot \left(1 - \alpha \left(\frac{2}{d-1} \log \frac{32}{(d-1)^2 \sigma_2} - \frac{4}{d-1} \log n + \frac{4}{d-1} \mathrm{l}_2 n + \log t \right) \right)^{-3} \\
=\ & C \cdot \left(\frac{32}{(d-1)^2 \sigma_2} \right)^2 \cdot \frac{\log n}{n} \cdot t^{d-1} \cdot a^3 \cdot \left(\frac{1}{\log n} \right. \\
& \left. - \frac{2\alpha}{(d-1)\log n} \log \frac{32}{(d-1)^2 \sigma_2} - \frac{4\alpha}{d-1} + \frac{4\alpha \mathrm{l}_2 n}{(d-1)\log n} + \frac{\alpha \log t}{\log n} \right)^{-3} \\
\to\ & 0,
\end{aligned}
$$

where asymptotic equality and convergence refer to $n \to \infty$. □

As expected, the probability for an exceedance of the distance over a threshold is asymptotically greater than the corresponding probability in case of a power type distribution. Indeed, the rescaling factor in the limit law of D_n in this case is of the order $O\left(n^{4/(d-1)}(\log n)^{-4/(d-1)}\right)$, which grows asymptotically faster than the rescaling factor of the order $O\left(n^{4/(d-1+4\alpha)}\right)$ in Section 3.3.

3.5 Exponential type

Throughout this section we suppose that the distribution function F of $Y_1 = 1 - R_1$ is differentiable in a small right neighborhood of 0, and that

$$F'(s) \sim \frac{a\alpha}{s^2} \exp\left(-\frac{\alpha}{s}\right)$$

as $s \downarrow 0$ for some $a > 0$ and $\alpha > 0$. By Lemma 3.3, we then have

$$F(s) \sim \int_0^s \frac{a\alpha}{t^2} \exp\left(-\frac{\alpha}{t}\right) dt = a \cdot \exp\left(-\frac{\alpha}{s}\right) \tag{3.23}$$

as $s \downarrow 0$. The right-hand continuity of F at 0 ensures that R_1 has the distribution function $1 - F(1-s)$ for sufficiently large $s \in [0,1]$. Thus, the distribution of R_1 is tail-equivalent to the exponential distribution function $1 - \exp\left(-\frac{\alpha}{1-t}\right) =: \Psi(t)$, i.e., we have

$$\lim_{t \uparrow 1} \frac{\mathbb{P}(R_1 > t)}{1 - \Psi(t)} = \lim_{s \downarrow 0} F(s) \cdot \exp\left(\frac{\alpha}{s}\right) = a > 0.$$

We therefore say that the distribution of R_1 is of exponential type.

Notice that the distribution function F of exponential type is rapidly varying with index ∞, i.e., we have

$$\lim_{s \downarrow 0} \frac{F(st)}{F(s)} = \lim_{s \downarrow 0} \exp\left(-\frac{\alpha}{st}(1-t)\right) \longrightarrow \begin{cases} 0, & \text{if } 0 < t < 1, \\ \infty, & \text{if } t > 1. \end{cases}$$

The distribution function F is differentiable in a right neighborhood of 0 and the derivative is positive and converges to 0 as $s \downarrow 0$. Moreover, F' decreases more slowly than each of the power distribution functions with $\alpha > 0$ (see page 23) when approaching 0, see Figure 3.4. Compared to the situation where F is of power type (see Section 3.3), it is less probable that random points appear in a narrow annulus close to the boundary of the unit ball. Hence, the probability of an exceedance is asymptotically smaller.

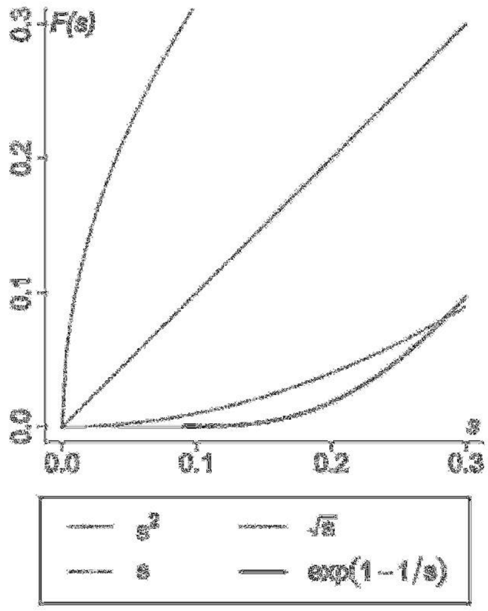

Figure 3.4: The function $\exp(1-1/s)$ increases more slowly than all power type distribution functions as $s \to 0$.

As before, we first determine the asymptotic behavior of $F * F(s)$ as $s \to 0$. By (3.23) and Lemma 3.3, substituting $u = t/s$ leads to the following asymptotic expression as $s \to 0$:

$$
\begin{aligned}
F * F(s) &\sim \int_{t=0}^{s} a\exp\left(-\frac{\alpha}{s-t}\right)\cdot\frac{a\alpha}{t^2}\exp\left(-\frac{\alpha}{t}\right)dt \\
&= \frac{a^2\alpha}{s}\int_{u=0}^{1}\exp\left(-\frac{\alpha}{s(1-u)u}\right)\frac{1}{u^2}\,du.
\end{aligned}
$$

Since the function $u \mapsto \frac{1}{(1-u)u}$ is decreasing on $(0,\frac{1}{2})$ and increasing on $(\frac{1}{2},1)$, we split up the integral into two parts and put $v = \frac{1}{(1-u)u}$, which implies $u = \frac{1}{2}\left(1 \mp \sqrt{1-\frac{4}{v}}\right)$ and $\frac{du}{dv} = \mp 1/\sqrt{v^3(v-4)}$, respectively. Consequently,

$$
\begin{aligned}
F * F(s) &\sim \frac{a^2\alpha}{s}\left[\int_{v=4}^{\infty}\exp\left(-\frac{\alpha}{s}v\right)4\left(1-\sqrt{1-\frac{4}{v}}\right)^{-2}\frac{1}{\sqrt{v^3(v-4)}}\,dv\right. \\
&\qquad\left.+\int_{v=4}^{\infty}\exp\left(-\frac{\alpha}{s}v\right)4\left(1+\sqrt{1-\frac{4}{v}}\right)^{-2}\frac{1}{\sqrt{v^3(v-4)}}\,dv\right] \\
&= \frac{a^2\alpha}{s}\int_{v=4}^{\infty}\exp\left(-\frac{\alpha}{s}v\right)\cdot\frac{v^2-2v}{\sqrt{v^3(v-4)}}\,dv.
\end{aligned}
$$

Finally, by substituting $w = \frac{\alpha}{s}(v-4)$, we have

$$
\begin{aligned}
F * F(s) &\sim a^2 s^{-\frac{1}{2}} \exp\left(-\frac{4\alpha}{s}\right) \int_{w=0}^{\infty} e^{-w}(sw+4\alpha)^{-\frac{1}{2}}(sw+2\alpha)w^{-\frac{1}{2}}\,dw \\
&\sim a^2 s^{-\frac{1}{2}} \exp\left(-\frac{4\alpha}{s}\right)(4\alpha)^{-\frac{1}{2}}(2\alpha) \int_{w=0}^{\infty} e^{-w} w^{-\frac{1}{2}}\,dw \\
&= \sqrt{\pi\alpha}\, a^2 s^{-\frac{1}{2}} \exp\left(-\frac{4\alpha}{s}\right), \qquad s \to 0,
\end{aligned} \tag{3.24}
$$

where the second asymptotic equivalence follows from the dominated convergence theorem.

Recall that the main idea for deriving the limit law of D_n is to determine the limit law of the random variable

$$
T_n := \sum_{1 \le i < j \le n} \mathbf{1}\left(\|X_i - X_j\| > 2 - \varepsilon_n\right),
$$

which counts the number of exceedances of the distance between two points over the threshold $2 - \varepsilon_n$. Condition (3.1) implies that the expected value of T_n converges to a positive finite constant, which entails the tightness of the sequence $(T_n)_{n \in \mathbb{N}}$. We first prove this convergence of $\mathbb{E}(T_n)$ for a suitable ε_n.

To this end, we first derive the asymptotic behavior of the lower and upper bound $p_1(\varepsilon)$, $p_2(\varepsilon)$, defined in (3.8) and (3.9), of the tail probability $p(\varepsilon) = \mathbb{P}(\|X_1 - X_2\| \geq 2 - \varepsilon)$ as $\varepsilon \to 0$. Using the same methods as in Proposition 3.10, applying (3.24) for the convolution of F and substituting $t = z/(\varepsilon - \frac{7}{4}\varepsilon^2)$, we have for $\varepsilon \to 0$

$$
\begin{aligned}
p_1(\varepsilon) &\sim \beta \cdot 4^{\frac{d-1}{2}} \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \frac{d-1}{2}\left(\varepsilon - \frac{7}{4}\varepsilon^2 - z\right)^{\frac{d-3}{2}} F * F(z)\,dz \\
&\sim \beta \cdot 4^{\frac{d-1}{2}} \sqrt{\pi\alpha}\, a^2 \frac{d-1}{2} \int_{z=0}^{\varepsilon - \frac{7}{4}\varepsilon^2} \left(\varepsilon - \frac{7}{4}\varepsilon^2 - z\right)^{\frac{d-3}{2}} z^{-\frac{1}{2}} \exp\left(-\frac{4\alpha}{z}\right) dz \\
&= \beta 4^{\frac{d-1}{2}} \sqrt{\pi\alpha}\, a^2 \frac{d-1}{2}\left(\varepsilon - \tfrac{7}{4}\varepsilon^2\right)^{\frac{d-2}{2}} \int_{t=0}^{1} (1-t)^{\frac{d-3}{2}} t^{-\frac{1}{2}} \exp\left(-\frac{4\alpha}{(\varepsilon - \frac{7}{4}\varepsilon^2)t}\right) dt
\end{aligned} \tag{3.25}
$$

with β given in (3.5). We now deal with the integral in (3.25). Putting $\kappa_\varepsilon =$

$4\alpha/(\varepsilon - \frac{7}{4}\varepsilon^2)$ for short and substituting $u = \kappa_\varepsilon/t - \kappa_\varepsilon$, we obtain

$$
\begin{aligned}
&\int_{t=0}^{1}(1-t)^{\frac{d-3}{2}}t^{-\frac{1}{2}}\exp\left(-\frac{4\alpha}{(\varepsilon-\frac{7}{4}\varepsilon^2)t}\right)dt \\
&= \int_{u=0}^{\infty}\left(\frac{u}{u+\kappa_\varepsilon}\right)^{\frac{d-3}{2}}\left(\frac{\kappa_\varepsilon}{u+\kappa_\varepsilon}\right)^{-\frac{1}{2}} e^{-u}e^{-\kappa_\varepsilon}\frac{\kappa_\varepsilon}{(u+\kappa_\varepsilon)^2}\,du \\
&= \kappa_\varepsilon^{\frac{1}{2}}e^{-\kappa_\varepsilon}\int_{u=0}^{\infty}u^{\frac{d-3}{2}}e^{-u}(u+\kappa_\varepsilon)^{-\frac{d}{2}}\,du \\
&\sim \kappa_\varepsilon^{-\frac{d-1}{2}}e^{-\kappa_\varepsilon}\int_{u=0}^{\infty}u^{\frac{d-3}{2}}e^{-u}du \\
&= \Gamma\left(\frac{d-1}{2}\right)\left(\frac{4\alpha}{\varepsilon-\frac{7}{4}\varepsilon^2}\right)^{-\frac{d-1}{2}}\exp\left(-\frac{4\alpha}{\varepsilon-\frac{7}{4}\varepsilon^2}\right),
\end{aligned}
$$

where the asymptotic equivalence above as $\varepsilon \to 0$ follows from the dominated convergence theorem. Plugging the asymptotic expression of the integral and formula (3.5) for β into (3.25), we have

$$p_1(\varepsilon) \sim \sigma_3 \cdot (\varepsilon - \tfrac{7}{4}\varepsilon^2)^{d-\frac{3}{2}} \cdot \exp\left(-\frac{4\alpha}{\varepsilon-\frac{7}{4}\varepsilon^2}\right)$$

as $\varepsilon \to 0$, where

$$\sigma_3 = \pi^{\frac{d}{2}}a^2\alpha^{-\frac{d-2}{2}}\int_{\mathbb{S}^{d-1}} g(u)g(-u)\mu^{d-1}(du).$$

Analogously, the upper bound $p_2(\varepsilon)$ is asymptotically equal to

$$p_2(\varepsilon) \sim \sigma_3 \cdot (\varepsilon + 5\varepsilon^2)^{d-\frac{3}{2}} \cdot \exp\left(-\frac{4\alpha}{\varepsilon+5\varepsilon^2}\right), \qquad \varepsilon \to 0.$$

Since

$$\lim_{\varepsilon\to 0}\frac{p_2(\varepsilon)}{p_1(\varepsilon)} = \lim_{\varepsilon\to 0}\left(\frac{\varepsilon+5\varepsilon^2}{\varepsilon-\frac{7}{4}\varepsilon^2}\right)^{d-\frac{3}{2}}\exp\left(\frac{4\alpha}{\varepsilon-\frac{7}{4}\varepsilon^2}-\frac{4\alpha}{\varepsilon+5\varepsilon^2}\right) = e^{27\alpha} > 1,$$

we do not know the exact asymptotic behavior of $p(\varepsilon)$, but we can conclude that the convergence of $p(\varepsilon)$ to 0 is of the order $O\left(\varepsilon^{d-\frac{3}{2}}\exp(-\frac{4\alpha}{\varepsilon})\right)$.

In the following, fix $t > 0$ and put

$$\varepsilon_n = \varepsilon_n(t) := \frac{2\alpha}{\gamma+\log n}(1+\delta_n) \tag{3.26}$$

with

$$\gamma := \frac{1}{2}\log\frac{\sigma_3(2\alpha)^{d-3/2}}{2t}, \qquad \delta_n := \frac{(d-3/2)\mathrm{l}_2 n}{2\log n}.$$

Since $\mathbb{E}(T_n) = \binom{n}{2}p(\varepsilon_n)$ and $p_1(\varepsilon_n) \le p(\varepsilon_n) \le p_2(\varepsilon_n)$, we now study the convergence of $q_1(n) := \log\left(\frac{n^2}{2}p_1(\varepsilon_n)\right)$ and $q_2(n) := \log\left(\frac{n^2}{2}p_2(\varepsilon_n)\right)$.

By plugging the Taylor series expansion of $1/(1-\frac{7}{4}\varepsilon_n) = 1 + \frac{7}{4}\varepsilon_n + O\left(\varepsilon_n^2\right)$ into $q_1(n)$, we have

$$\begin{aligned}
q_1(n) &= \log\frac{\sigma_3}{2} + 2\log n - \frac{4\alpha}{\varepsilon_n - \frac{7}{4}\varepsilon_n^2} + (d-\tfrac{3}{2})\log(\varepsilon_n - \tfrac{7}{4}\varepsilon_n^2)\\
&= \log\frac{\sigma_3}{2} + 2\log n - \frac{4\alpha}{\varepsilon_n}\cdot\frac{1}{1-\frac{7}{4}\varepsilon_n} + (d-\tfrac{3}{2})\log\varepsilon_n + (d-\tfrac{3}{2})\log(1-\tfrac{7}{4}\varepsilon_n)\\
&= \log\frac{\sigma_3}{2} + 2\log n - \frac{4\alpha}{\varepsilon_n} - 7\alpha + (d-\tfrac{3}{2})\log\varepsilon_n + o(1)\\
&= \log\frac{\sigma_3}{2} + 2\log n - \frac{2(\gamma+\log n)}{1+\delta_n} - 7\alpha\\
&\quad + (d-\tfrac{3}{2})\left[\log(2\alpha) - \log(\gamma+\log n) + \log(1+\delta_n)\right] + o(1).
\end{aligned}$$

Using $1/(1+\delta_n) = 1-\delta_n+O\left(\delta_n^2\right)$ and $\log(\gamma+\log n) = \log(\log n(1+\gamma/\log n)) = \mathrm{l}_2 n + o(1)$, we obtain

$$\begin{aligned}
q_1(n) &= \log\frac{\sigma_3}{2} + 2\log n - 2\gamma - 2\log n + 2\delta_n\log n - 7\alpha\\
&\quad + (d-\tfrac{3}{2})\log(2\alpha) - (d-\tfrac{3}{2})\mathrm{l}_2 n + o(1).
\end{aligned}$$

Plugging γ and δ_n into $q_1(n)$, we have

$$\begin{aligned}
q_1(n) &= \log\frac{\sigma_3}{2} - \log\frac{\sigma_3(2\alpha)^{d-3/2}}{2t} + (d-\tfrac{3}{2})\mathrm{l}_2 n - 7\alpha\\
&\quad + (d-\tfrac{3}{2})\log(2\alpha) - (d-\tfrac{3}{2})\mathrm{l}_2 n + o(1)\\
&= \log t - 7\alpha + o(1).
\end{aligned}$$

Analogously, we have

$$q_2(n) = \log t + 20\alpha + o(1).$$

Consequently, the lower and the upper bound of the expected value of T_n converge both to some (diff erent) nite positive constants, i.e., we have

$$\lim_n \frac{n^2}{2}p_1(\varepsilon_n) = t\cdot e^{-7\alpha}, \qquad \lim_n \frac{n^2}{2}p_2(\varepsilon_n) = t\cdot e^{20\alpha}$$

and thus

$$t \cdot e^{-7\alpha} \le \liminf_n \mathbb{E}(T_n) \le \limsup_n \mathbb{E}(T_n) \le t \cdot e^{20\alpha}.$$

Hence, the sequence $(T_n)_{n\in\mathbb{N}}$ is tight.

On the other hand, we will prove that for ε_n given in (3.26) condition (3.2) is not satisfied. To be more specific, the probability for two overlapping exceedances rescaled by n^3 converges to infinity as $n \to \infty$. In view of Chapter 2, this means that the variance of T_n converges to infinity.

In the following, we use again the geometric considerations and asymptotic methods of Section 3.2 and Section 3.3. Set $\tilde{\varepsilon}_n = \varepsilon_n - \frac{7}{4}\varepsilon_n^2$ for short, the substitution $t_1 = y_1/\tilde{\varepsilon}_n$ and $t_2 = y_2/(\tilde{\varepsilon}_n - y_1) = y_2/\tilde{\varepsilon}_n(1-t_1)$ yield

$$\begin{aligned}
& n^3 \mathbb{P}\left(\|X_1 - X_2\| \ge 2 - \varepsilon_n,\ \|X_1 - X_3\| \ge 2 - \varepsilon_n\right) \\
\ge\ & n^3 \mathbb{P}\Big(\phi \le \left(4(\varepsilon_n - Y_1 - Y_2) - 7\varepsilon_n^2\right)^{\frac{1}{2}},\ Y_1 + Y_2 \le \varepsilon_n - \tfrac{7}{4}\varepsilon_n^2 \\
& \qquad \phi \le \left(4(\varepsilon_n - Y_1 - Y_3) - 7\varepsilon_n^2\right)^{\frac{1}{2}},\ Y_1 + Y_3 \le \varepsilon_n - \tfrac{7}{4}\varepsilon_n^2\Big) \\
=\ & n^3 \int_{y_1=0}^{\tilde{\varepsilon}_n} \left(\int_{y_2=0}^{\tilde{\varepsilon}_n - y_1} \mathbb{P}\Big(\phi \le \left(4(\varepsilon_n - y_1 - y_2) - 7\varepsilon_n^2\right)^{\frac{1}{2}}\Big)\, dF(y_2) \right)^2 dF(y_1) \\
=\ & n^3 \int_{t_1=0}^{1} \left(\int_{t_2=0}^{1} \mathbb{P}\Big(\phi \le [4\tilde{\varepsilon}_n(1-t_1)(1-t_2)]^{\frac{1}{2}}\Big)\, dF(\tilde{\varepsilon}_n(1-t_1)t_2) \right)^2 dF(\tilde{\varepsilon}_n t_1) \\
=:\ & n^3\, p_3(n). \qquad (3.27)
\end{aligned}$$

Applying Lemma 3.4 and integration by parts, the inner integral in (3.27) is then asymptotically equivalent to

$$\begin{aligned}
& \int_{t_2=0}^{1} \beta 4^{\frac{d-1}{2}} \tilde{\varepsilon}_n^{\frac{d-1}{2}} (1-t_1)^{\frac{d-1}{2}} (1-t_2)^{\frac{d-1}{2}} dF(\tilde{\varepsilon}_n(1-t_1)t_2) \\
=\ & \beta 4^{\frac{d-1}{2}} \tilde{\varepsilon}_n^{\frac{d-1}{2}} (1-t_1)^{\frac{d-1}{2}} \int_{t_2=0}^{1} \frac{d-1}{2} (1-t_2)^{\frac{d-3}{2}} F(\tilde{\varepsilon}_n(1-t_1)t_2)\, dt_2 \\
\sim\ & \beta 4^{\frac{d-1}{2}} a \frac{d-1}{2} \tilde{\varepsilon}_n^{\frac{d-1}{2}} (1-t_1)^{\frac{d-1}{2}} \int_{t_2=0}^{1} (1-t_2)^{\frac{d-3}{2}} \exp\left(-\frac{\alpha}{\tilde{\varepsilon}_n(1-t_1)t_2}\right) dt_2
\end{aligned} \qquad (3.28)$$

as $n \to 0$. Set $\kappa_\varepsilon(t_1) = \alpha/\tilde{\varepsilon}_n(1-t_1)$ and substitute $u = \kappa_\varepsilon(t_1)/t_2 - \kappa_\varepsilon(t_1)$. This gives

$$
\begin{aligned}
&\int_{t_2=0}^{1} (1-t_2)^{\frac{d-3}{2}} \exp\left(-\frac{\alpha}{\tilde{\varepsilon}_n(1-t_1)t_2}\right) dt_2 \\
&= \int_{u=0}^{\infty} \left(\frac{u}{\kappa_\varepsilon(t_1)+u}\right)^{\frac{d-3}{2}} e^{-u} e^{-\kappa_\varepsilon(t_1)} \frac{\kappa_\varepsilon(t_1)}{(\kappa_\varepsilon(t_1)+u)^2}\, du \\
&= e^{-\kappa_\varepsilon(t_1)} \kappa_\varepsilon(t_1)^{-\frac{d-1}{2}} \int_{u=0}^{\infty} u^{\frac{d-3}{2}} e^{-u} \left(\frac{\kappa_\varepsilon(t_1)}{\kappa_\varepsilon(t_1)+u}\right)^{\frac{d+1}{2}} du \\
&\sim \Gamma\left(\frac{d-1}{2}\right) e^{-\kappa_\varepsilon(t_1)} \kappa_\varepsilon(t_1)^{-\frac{d-1}{2}},
\end{aligned}
$$

where "$\sim$" refers to $n \to 0$ and follows from the dominated convergence theorem. Putting this into (3.28) and then into (3.27), integration by parts yields

$$
\begin{aligned}
&n^3 p_3(n) \\
&\sim C_1 n^3 \tilde{\varepsilon}_n^{2d-2} \int_{t_1=0}^{1} (1-t_1)^{2d-2} \exp\left(-\frac{2\alpha}{\tilde{\varepsilon}_n(1-t_1)}\right) dF(\tilde{\varepsilon}_n t_1) \\
&\sim C_1 n^3 \tilde{\varepsilon}_n^{2d-2} \left(2a(d-1) \int_{t_1=0}^{1} (1-t_1)^{2d-3} \exp\left(-\frac{\alpha(1+t_1)}{\tilde{\varepsilon}_n(1-t_1)t_1}\right) dt_1 \right. \\
&\qquad \left. + \frac{2a\alpha}{\tilde{\varepsilon}_n} \int_{t_1=0}^{1} (1-t_1)^{2d-4} \exp\left(-\frac{\alpha(1+t_1)}{\tilde{\varepsilon}_n(1-t_1)t_1}\right) dt_1 \right) \qquad (3.29)
\end{aligned}
$$

as $n \to 0$, where C_1 is some positive finite constant. Since the two integrals in (3.29) have the same structure, we compute in the following the integral

$$
\int_{t_1=0}^{1} (1-t_1)^k \exp\left(-\frac{\alpha(1+t_1)}{\tilde{\varepsilon}_n(1-t_1)t_1}\right) dt_1 \qquad (3.30)
$$

for a general $k \geq 0$. By studying the function $v : t_1 \mapsto \frac{1+t_1}{(1-t)t_1}$ on the interval $(0,1)$ we find that v has the local minimum $\frac{\sqrt{2}}{3\sqrt{2}-4} =: c$ at $t_1 = \sqrt{2}-1$ and the function is decreasing on $(0, \sqrt{2}-1)$ and increasing on $(\sqrt{2}-1, 1)$. Moreover, the inverse function on the two interval parts is $t_1 = (v - 1 \mp \sqrt{1-6v-v^2})/2v$ with $\frac{dt_1}{dv} = \mp (3v-1)/(2v^2\sqrt{1-6v+v^2}) + \frac{1}{2v^2}$, respectively. According to this consideration, we separate the interval of integration $(0,1)$ into two parts and substitute $v = \frac{1+t_1}{(1-t_1)t_1}$. Then (3.30) is equal to

$$\int_{t_1=0}^{\sqrt{2}-1} (1-t_1)^k \exp\left(-\tfrac{\alpha(1+t_1)}{\tilde{\varepsilon}_n(1-t_1)t_1}\right) dt_1 + \int_{t_1=\sqrt{2}-1}^{1} (1-t_1)^k \exp\left(-\tfrac{\alpha(1+t_1)}{\tilde{\varepsilon}_n(1-t_1)t_1}\right) dt_1$$

$$= \int_{v=c}^{\infty} \exp\left(-\tfrac{\alpha}{\tilde{\varepsilon}_n} v\right) \left[\left(\frac{v+1+\sqrt{1-6v+v^2}}{2v}\right)^k \left(\frac{3v-1}{2v^2\sqrt{1-6v+v^2}} - \frac{1}{2v^2}\right)\right.$$
$$\left. + \left(\frac{v+1-\sqrt{1-6v+v^2}}{2v}\right)^k \left(\frac{3v-1}{2v^2\sqrt{1-6v+v^2}} + \frac{1}{2v^2}\right)\right] dv. \tag{3.31}$$

Furthermore, we use the substitution $w = \frac{\alpha}{\tilde{\varepsilon}_n}(v-c)$. For a fixed $v \in (0,\infty)$ the terms in (3.31) reduce to

$$\left(\frac{v+1+\sqrt{1-6v+v^2}}{2v}\right)^k = \frac{c+1}{2c} + O\left(\tilde{\varepsilon}_n^2\right),$$
$$\left(\frac{v+1-\sqrt{1-6v+v^2}}{2v}\right)^k = \frac{c+1}{2c} + O\left(\tilde{\varepsilon}_n^2\right)$$

and

$$\frac{3v-1}{2v^2\sqrt{1-6v+v^2}} - \frac{1}{2v^2} = \frac{3c-1}{2c^2\sqrt{2c-6}}\sqrt{\frac{\alpha}{\tilde{\varepsilon}_n w}} - \frac{1}{2c^2} + O\left(\tilde{\varepsilon}_n^{\frac{1}{2}}\right),$$
$$\frac{3v-1}{2v^2\sqrt{1-6v+v^2}} + \frac{1}{2v^2} = \frac{3c-1}{2c^2\sqrt{2c-6}}\sqrt{\frac{\alpha}{\tilde{\varepsilon}_n w}} + \frac{1}{2c^2} + O\left(\tilde{\varepsilon}_n^{\frac{1}{2}}\right)$$

as $\tilde{\varepsilon}_n \to 0$. Then, invoking (3.31) and (3.30), we have

$$\int_{t_1=0}^{1} (1-t_1)^k \exp\left(-\frac{\alpha(1+t_1)}{\tilde{\varepsilon}_n(1-t_1)t_1}\right) dt_1$$
$$\sim \exp\left(-\frac{c\alpha}{\tilde{\varepsilon}_n}\right)\left(\frac{c+1}{2c}\right)^k \frac{2(3c-1)}{2c^2\sqrt{2c-6}}\sqrt{\frac{\alpha}{\tilde{\varepsilon}_n}} \cdot \frac{\tilde{\varepsilon}_n}{\alpha}\int_{w=0}^{\infty} e^{-w} w^{-\frac{1}{2}}\, dw$$
$$= C_2\, \tilde{\varepsilon}_n^{\frac{1}{2}} \exp\left(-\frac{c\alpha}{\tilde{\varepsilon}_n}\right)$$

as $n \to \infty$, where C_2 is some positive finite constant. Using this on the integrals in (3.29), we then obtain

$$
\begin{aligned}
n^3\, p_3(n) \;&\sim\; C_1 n^3 \tilde{\varepsilon}_n^{2d-2}\;\; C_3 \tilde{\varepsilon}_n^{\frac{1}{2}} \exp\; -\frac{c\alpha}{\tilde{\varepsilon}_n} \;+ C_4 \tilde{\varepsilon}_n^{-\frac{1}{2}} \exp\; -\frac{c\alpha}{\tilde{\varepsilon}_n} \\
&\sim\; C_1 C_4\, n^3 \left(\varepsilon_n - \tfrac{7}{4}\varepsilon_n^2\right)^{2d-\frac{5}{2}} \exp\; -\frac{c\alpha}{\varepsilon_n - \frac{7}{4}\varepsilon_n^2}
\end{aligned}
\tag{3.32}
$$

as n , where C_1, C_3 and C_4 are positive finite constants.

To see the divergence of $n^3 p_3(n)$, we now study $\log(n^3 p_3(n))$. Plugging ε_n given in (3.26) into (3.32), we have

$$
\begin{aligned}
&\log(n^3 p_3(n)) \\
=\;& \log(C_1C_4) + 3\log n - \frac{c\alpha}{\varepsilon_n - \frac{7}{4}\varepsilon_n^2} + (2d - \tfrac{5}{2})\log(\varepsilon_n - \tfrac{7}{4}\varepsilon_n^2) \\
=\;& \log(C_1C_4) + 3\log n - \frac{c\alpha}{\varepsilon_n} - \frac{7}{4}c\alpha + (2d - \tfrac{5}{2})\log \varepsilon_n + o(1) \\
=\;& \log(C_1C_4) + 3\log n - \frac{c(\gamma + \log n)}{2(1+\delta_n)} - \frac{7}{4}c\alpha \\
&+ (2d - \tfrac{5}{2})\left[\log(2\alpha) - \log(\gamma + \log n) + \log(1+\delta_n)\right] + o(1) \\
=\;& C_5 + \left(3 - \frac{c}{2}\right)\log n - \;\; (2 - \frac{c}{4})d + \frac{3c}{8} - \frac{5}{2} \;\; \mathrm{l}_2 n + o(1),
\end{aligned}
$$

where C_5 is a positive finite constant. Since $3 - c/2 = 3 - \frac{\sqrt{2}}{2(3\sqrt{2}-4)} \approx 3 - 5.828/2 > 0$, we have

$$
\log(n^3 p_3(n)) - \qquad , \qquad n \qquad ,
$$

which implies the divergence of the expression in condition (3.2) as well as that of $\mathbb{V}(T_n)$.

Since condition (3.2) is not satisfied with ε_n given in (3.26), we cannot use Lemma 3.1 to obtain the limit law of D_n, although condition (3.1) holds for some subsequence $(\mathbb{E}(T_{n_k}))_{k\geq 1}$. Furthermore, we will show in the following that the random variable T_n converges in probability to zero as n tends to infinity.

Set

$$
\hat{\varepsilon}_n = \hat{\varepsilon}_n(t) := \frac{\varepsilon_n + 5\varepsilon_n^2}{2}(1 - \nu_n)
$$

with ε_n given in (3.26) and $\nu_n := (d - 3/2)\mathrm{l}_2 n/\log n$. Then, $\hat{\varepsilon}_n < \varepsilon_n/2$ for sufficiently large n. Letting

$$\begin{aligned}
E(i,j) &= \{\|X_i - X_j\| \geq 2 - \varepsilon_n\}, \\
A(i) &= \{\|X_i\| \geq 1 - \hat{\varepsilon}_n\}, \\
B(i) &= \{1 - (\varepsilon_n - \hat{\varepsilon}_n) \leq \|X_i\| \leq 1 - \hat{\varepsilon}_n\},
\end{aligned}$$

it follows that

$$T_n = T_n^{(1)} + T_n^{(2)},$$

where

$$\begin{aligned}
T_n^{(1)} &:= \sum_{1 \leq i < j \leq n} \mathbf{1}\{E(i,j) \cap (A(i) \cup A(j))\}, \\
T_n^{(2)} &:= \sum_{1 \leq i < j \leq n} \mathbf{1}\{E(i,j) \cap B(i) \cap B(j)\}.
\end{aligned}$$

Since $\{T_n^{(1)} > 0\} \subset \cup_{i=1}^{n} A(i)$, we have for fixed $\epsilon > 0$

$$\begin{aligned}
\mathbb{P}\left(T_n^{(1)} > \epsilon\right) &\leq \sum_{i=1}^{n} \mathbb{P}(\|X_i\| \geq 1 - \hat{\varepsilon}_n) \\
&= n \cdot \mathbb{P}(1 - \|X_i\| \leq \hat{\varepsilon}_n) \\
&\sim n \cdot a \exp\left(-\frac{\alpha}{\hat{\varepsilon}_n}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
&\log n + \log a - \frac{\alpha}{\hat{\varepsilon}_n} \\
&= \log n + \log a - \left(\tfrac{2\alpha}{\varepsilon_n} - 10\alpha + o(1)\right)(1 - \nu_n)^{-1} \\
&= \log n + \log a - \left((\gamma + \log n)(1 + \delta_n)^{-1} - 10\alpha + o(1)\right)(1 - \nu_n)^{-1} \\
&= \log a - \gamma + 10\alpha - \left(\tfrac{d - 3/2}{2}\right)\mathrm{l}_2 n + o(1) \to -\infty
\end{aligned}$$

as $n \to \infty$, which means that $T_n^{(1)} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$.

To show the convergence of $T_n^{(2)}$ we need to study the modified tail probability $\mathbb{P}(E(1,2) \cap B(1) \cap B(2))$. Let $\bar{\varepsilon}_n = \varepsilon_n + 5\varepsilon_n^2$ for short. By adopting the notations and geometric considerations in Section 3.2, we have

$$
\begin{aligned}
& \mathbb{P}\,(E(1,2) \setminus B(1) \setminus B(2)) \\
\le\ & \mathbb{P}\Big(\quad \phi \ \ [4(\bar{\varepsilon}_n - Y_1 - Y_2) + 20\varepsilon_n^2]^{\frac{1}{2}}, Y_1 + Y_2 \le \varepsilon_n, \hat{\varepsilon}_n \le Y_1, Y_2 \le \varepsilon_n - \hat{\varepsilon}_n\Big) \\
\le\ & \int_{y_1=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n-\hat{\varepsilon}_n} \int_{y_2=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n - y_1} \mathbb{P}\Big(\quad \phi \ \ [4(\bar{\varepsilon}_n - y_1 - y_2)]^{\frac{1}{2}}\Big)\, dF(y_2)\, dF(y_1) \\
\sim\ & \beta\, 4^{\frac{d-1}{2}} \int_{y_1=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n-\hat{\varepsilon}_n} \int_{y_2=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n - y_1} (\bar{\varepsilon}_n - y_1 - y_2)^{\frac{d-1}{2}}\, dF(y_2)\, dF(y_1) \\
=\ & \beta\, 4^{\frac{d-1}{2}} \int_{y_1=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n-\hat{\varepsilon}_n} (\bar{\varepsilon}_n - y_1 - y_2)^{\frac{d-1}{2}} F(y_2)\Big|_{y_2=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n - y_1} \\
& \qquad\qquad + \int_{y_2=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n - y_1} \frac{d-1}{2}(\bar{\varepsilon}_n - y_1 - y_2)^{\frac{d-3}{2}} F(y_2)\, dy_2 \quad dF(y_1) \\
=:\ & \beta\, 4^{\frac{d-1}{2}} \cdot \{p_5(n) + p_6(n)\} \ =:\ p_4(n)
\end{aligned}
\tag{3.33}
$$

with

$$
\begin{aligned}
p_5(n) &:= \int_{y_1=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n-\hat{\varepsilon}_n} - (\bar{\varepsilon}_n - \hat{\varepsilon}_n - y_1)^{\frac{d-1}{2}} F(\hat{\varepsilon}_n)\, dF(y_1), \\
p_6(n) &:= \int_{y_1=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n-\hat{\varepsilon}_n} \int_{y_2=\hat{\varepsilon}_n}^{\bar{\varepsilon}_n - y_1} \frac{d-1}{2}(\bar{\varepsilon}_n - y_1 - y_2)^{\frac{d-3}{2}} F(y_2)\, dy_2\, dF(y_1),
\end{aligned}
$$

where "$\sim$ " in (3.33) refers to n and follows from Lemma 3.4. We now deal with $p_5(n)$ and $p_6(n)$, respectively.

Integration by parts, followed by the substitution $t = (y_1 - \hat{\varepsilon}_n)/(\bar{\varepsilon}_n - 2\hat{\varepsilon}_n)$ and invoking (3.23) lead to

$$
\begin{aligned}
p_5(n) \ \sim\ & (\bar{\varepsilon}_n - 2\hat{\varepsilon}_n)^{\frac{d-1}{2}} F(\hat{\varepsilon}_n)^2 - \frac{d-1}{2} a(\bar{\varepsilon}_n - 2\hat{\varepsilon}_n)^{\frac{d-1}{2}} F(\hat{\varepsilon}_n) \\
& \cdot \int_{t=0}^{1} (1-t)^{\frac{d-3}{2}} \exp\ -\frac{\alpha}{(\bar{\varepsilon}_n - 2\hat{\varepsilon}_n)t + \hat{\varepsilon}_n} \quad dt
\end{aligned}
$$

as $n \to \infty$. Letting $\kappa_n = \alpha/(\bar{\varepsilon}_n - \hat{\varepsilon}_n)$ and $u = \frac{\alpha}{(\alpha/\kappa_n - \hat{\varepsilon}_n)t + \hat{\varepsilon}_n} - \kappa_n$, we obtain

$$\begin{aligned} p_5(n) \ \sim\ & (\bar{\varepsilon}_n - 2\hat{\varepsilon}_n)^{\frac{d-1}{2}} F(\hat{\varepsilon}_n)^2 - \frac{d-1}{2} a\alpha^{-\frac{d-1}{2}} F(\hat{\varepsilon}_n)(\bar{\varepsilon}_n - \hat{\varepsilon}_n)^{d-1} \\ & \cdot \exp\left(-\frac{\alpha}{\bar{\varepsilon}_n - \hat{\varepsilon}_n}\right) \int_{u=0}^{\tau_n} u^{\frac{d-3}{2}} e^{-u} \left(\frac{\kappa_n}{u+\kappa_n}\right)^{\frac{d+1}{2}} du \end{aligned}$$

where $\tau_n = \alpha/\hat{\varepsilon}_n - \kappa_n = 4\alpha\nu_n/(\bar{\varepsilon}_n(1-\nu_n^2))$. Since for fixed $u \in (0, \infty)$ we have $\kappa_n/(u+\kappa_n) \to 1$ as $n \to \infty$ and $\frac{\kappa_n}{u+\kappa_n} \le 1$, then

$$\begin{aligned} \int_{u=0}^{\tau_n} u^{\frac{d-3}{2}} e^{-u} \left(\frac{\kappa_n}{u+\kappa_n}\right)^{\frac{d+1}{2}} du \ &\le\ \int_{u=0}^{\infty} u^{\frac{d-3}{2}} e^{-u} \left(\frac{\kappa_n}{u+\kappa_n}\right)^{\frac{d+1}{2}} du \\ &\to\ \int_{u=0}^{\infty} u^{\frac{d-3}{2}} e^{-u} du \ =\ \Gamma\left(\frac{d-1}{2}\right) \end{aligned}$$

by the dominated convergence theorem. On the other hand, since $\tau_n \to \infty$ as $n \to \infty$, we have for fixed $c > 0$ and sufficiently large n

$$\int_{u=0}^{\tau_n} u^{\frac{d-3}{2}} e^{-u} \left(\frac{\kappa_n}{u+\kappa_n}\right)^{\frac{d+1}{2}} du \ \ge\ \int_{u=0}^{c} u^{\frac{d-3}{2}} e^{-u} \left(\frac{\kappa_n}{u+\kappa_n}\right)^{\frac{d+1}{2}} du.$$

As $n \to \infty$, the latter integral converges to $\int_{u=0}^{c} u^{\frac{d-3}{2}} e^{-u} du$ by the dominated convergence theorem. Letting $c \to \infty$, it follows that

$$\lim_{n\to\infty} \int_{u=0}^{\tau_n} u^{\frac{d-3}{2}} e^{-u} \left(\frac{\kappa_n}{u+\kappa_n}\right)^{\frac{d+1}{2}} du = \Gamma\left(\frac{d-1}{2}\right).$$

Hence,

$$\begin{aligned} p_5(n) \ \sim\ & a^2 \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n(1-\nu_n)}\right) (\bar{\varepsilon}_n \nu_n)^{\frac{d-1}{2}} \\ & - a^2 \alpha^{-\frac{d-1}{2}} \Gamma\left(\tfrac{d+1}{2}\right) 2^{-d+1} \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n(1-\nu_n^2)}\right) \bar{\varepsilon}_n^{d-1}(1+\nu_n)^{d-1} \end{aligned}$$

as $n \to \infty$.

By using the substitution $t_1 = y_1/\bar{\varepsilon}_n$ and $t_2 = (y_2 - \hat{\varepsilon}_n)/(\bar{\varepsilon}_n - \hat{\varepsilon}_n - y_1)$ and (3.23), it follows that

$$\begin{aligned} p_6(n) \ \sim\ & \frac{d-1}{2} a \int_{t_1=(1-\nu_n)/2}^{(1+\nu_n)/2} (\bar{\varepsilon}_n - \hat{\varepsilon}_n - \bar{\varepsilon}_n t_1)^{\frac{d-1}{2}} \int_{t_2=0}^{1} (1-t_2)^{\frac{d-3}{2}} \\ & \cdot \exp\left(-\frac{\alpha}{(\bar{\varepsilon}_n - \hat{\varepsilon}_n - \bar{\varepsilon}_n t_1)t_2 + \hat{\varepsilon}_n}\right) dt_2 \, dF(\bar{\varepsilon}_n t_1). \end{aligned}$$

as $n \to \infty$. Set $\kappa_n(t_1) := \alpha/(\bar{\varepsilon}_n(1-t_1))$ and $u := \frac{\alpha}{(\alpha/\kappa_n(t_1)-\hat{\varepsilon}_n)t_2+\hat{\varepsilon}_n} - \kappa_n(t_1)$, we then have

$$p_6(n) \sim \frac{d-1}{2}\, a\alpha^{-\frac{d-1}{2}}\bar{\varepsilon}_n^{d-1} \int_{t_1=(1-\nu_n)/2}^{(1+\nu_n)/2} (1-t_1)^{d-1}\exp\left(-\frac{\alpha}{\bar{\varepsilon}_n(1-t_1)}\right) \int_{u=0}^{\tau_n(t_1)} u^{\frac{d-3}{2}}e^{-u}\left(\frac{\kappa_n(t_1)}{u+\kappa_n(t_1)}\right)^{\frac{d+1}{2}} du\, dF(\bar{\varepsilon}_n t_1)$$

with $\tau_n(t_1) = \alpha/\hat{\varepsilon}_n - \kappa_n(t_1)$. By the dominated convergence theorem, we obtain on one hand

$$\begin{aligned} &\int_{u=0}^{\tau_n(t_1)} u^{\frac{d-3}{2}}e^{-u}\left(\frac{\kappa_n(t_1)}{u+\kappa_n(t_1)}\right)^{\frac{d+1}{2}} du \\ \leq &\int_{u=0}^{\infty} u^{\frac{d-3}{2}}e^{-u}\left(\frac{\kappa_n(t_1)}{u+\kappa_n(t_1)}\right)^{\frac{d+1}{2}} du \\ \to &\int_{0}^{\infty} u^{\frac{d-3}{2}}e^{-u}du = \Gamma\left(\frac{d-1}{2}\right). \end{aligned}$$

On the other hand, we have for fixed $c > 0$ and sufficiently large n

$$\begin{aligned} &\int_{u=0}^{\tau_n(t_1)} u^{\frac{d-3}{2}}e^{-u}\left(\frac{\kappa_n(t_1)}{u+\kappa_n(t_1)}\right)^{\frac{d+1}{2}} du \\ \geq &\int_{u=0}^{c} u^{\frac{d-3}{2}}e^{-u}\left(\frac{\kappa_n(t_1)}{u+\kappa_n(t_1)}\right)^{\frac{d+1}{2}} du. \end{aligned}$$

The latter integral converges to $\int_0^c u^{\frac{d-3}{2}}e^{-u}du$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \int_{u=0}^{\tau_n(t_1)} u^{\frac{d-3}{2}}e^{-u}\left(\frac{\kappa_n(t_1)}{u+\kappa_n(t_1)}\right)^{\frac{d+1}{2}} du = \Gamma\left(\frac{d-1}{2}\right).$$

By using integration by parts and (3.23) again, we have

$$
\begin{aligned}
&p_6(n)\\
&\sim a\alpha^{-\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)\bar{\varepsilon}_n^{d-1}\int_{t_1=(1-\nu_n)/2}^{(1+\nu_n)/2}(1-t_1)^{d-1}\exp\left(-\frac{\alpha}{\bar{\varepsilon}_n(1-t_1)}\right)dF(\bar{\varepsilon}_n t_1)\\
&\sim a\alpha^{-\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)\bar{\varepsilon}_n^{d-1}\Bigg(a\left[\left(\tfrac{1-\nu_n}{2}\right)^{d-1}-\left(\tfrac{1+\nu_n}{2}\right)^{d-1}\right]\exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n(1-\nu_n^2)}\right)\\
&\qquad+\frac{a\alpha}{\bar{\varepsilon}_n}\int_{t_1=(1-\nu_n)/2}^{(1+\nu_n)/2}(1-t_1)^{d-3}\exp\left(-\frac{\alpha}{\bar{\varepsilon}_n t_1(1-t_1)}\right)dt_1\Bigg)
\end{aligned}
\tag{3.34}
$$

as $n\to\infty$, where the derivative of $(1-t_1)^{d-1}\exp\left(-\frac{\alpha}{\bar{\varepsilon}_n(1-t_1)}\right)$ is asymptotically equal to $\frac{\alpha}{\bar{\varepsilon}_n}(1-t_1)^{d-3}$ as $n\to\infty$ by omitting the asymptotically negligible terms.

To compute the last integral, we substitute $v:=\frac{1}{t_1(1-t_1)}$. Since this function has a local minimum at $1/2$, we split up the interval of integration into two parts $(\frac{1-\nu_n}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1+\nu_n}{2})$. The inverse function is $t_1=\frac{1}{2}\left(1\mp\sqrt{1-\frac{4}{v}}\right)$ and the derivative is $\frac{dt_1}{dv}=\mp 1/\sqrt{v^3(v-4)}$, respectively. It follows that

$$
\begin{aligned}
&\int_{t_1=(1-\nu_n)/2}^{(1+\nu_n)/2}(1-t_1)^{d-3}\exp\left(-\frac{\alpha}{\bar{\varepsilon}_n t_1(1-t_1)}\right)dt_1\\
&=\int_{v=4}^{4/(1-\nu_n^2)}\exp\left(-\frac{\alpha}{\bar{\varepsilon}_n}v\right)\frac{\left(1+\sqrt{1-4/v}\right)^{d-3}+\left(1-\sqrt{1-4/v}\right)^{d-3}}{2^{d-3}\sqrt{v^3(v-4)}}\,dv.
\end{aligned}
\tag{3.35}
$$

Substitute $w=\frac{\alpha}{\bar{\varepsilon}_n}(v-4)$, the integral is then equivalent to

$$
\begin{aligned}
&\int_{w=0}^{\rho_n}\exp\left(-w-\frac{4\alpha}{\bar{\varepsilon}_n}\right)2^{-d+3}\sqrt{\frac{\alpha\bar{\varepsilon}_n}{\bar{\varepsilon}_n w(\bar{\varepsilon}_n w+4\alpha)^3}}\\
&\qquad\cdot\left[\left(1+\sqrt{\frac{\bar{\varepsilon}_n w}{\bar{\varepsilon}_n w+4\alpha}}\right)^{d-3}+\left(1-\sqrt{\frac{\bar{\varepsilon}_n w}{\bar{\varepsilon}_n w+4\alpha}}\right)^{d-3}\right]dw\\
&=2^{-d+3}\alpha\bar{\varepsilon}_n^{\frac{1}{2}}\exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n}\right)\int_{w=0}^{\rho_n}e^{-w}w^{-\frac{1}{2}}\zeta_n(w)\,dw
\end{aligned}
\tag{3.36}
$$

where $\rho_n := 4\alpha\nu_n^2/(\bar{\varepsilon}_n(1-\nu_n^2)) \to 0$ as $n \to \infty$ and

$$\zeta_n(w) := (\bar{\varepsilon}_n w + 4\alpha)^{-\frac{3}{2}} \cdot \left[\left(1+\sqrt{\frac{\bar{\varepsilon}_n w}{\bar{\varepsilon}_n w + 4\alpha}}\right)^{d-3} + \left(1-\sqrt{\frac{\bar{\varepsilon}_n w}{\bar{\varepsilon}_n w + 4\alpha}}\right)^{d-3}\right].$$

To compute the integral in (3.36), we substitute $s := w/\rho_n$ and obtain

$$\int_{w=0}^{\rho_n} e^{-w} w^{-\frac{1}{2}}\, \zeta_n(w)\, dw = \rho_n^{\frac{1}{2}} \int_{s=0}^{1} s^{-\frac{1}{2}}\, e^{-\rho_n s}\, \zeta_n(\rho_n s)\, ds.$$

Since for each fixed $s \in (0,1)$ we have $\lim_{n\to\infty} e^{-\rho_n s} = 1$, $\lim_{n\to\infty} \zeta_n(\rho_n s) = 2^{-2}\alpha^{-3/2}$ and $e^{-\rho_n s}\zeta_n(\rho_n s)$ is uniformly bounded in n and s by $(4\alpha)^{-3/2}(2^{d-3}+1)$, it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int_{s=0}^{1} s^{-\frac{1}{2}} e^{-\rho_n s} \zeta_n(\rho_n s)\, ds \;=\; 2^{-2}\alpha^{-\frac{3}{2}} \int_{s=0}^{1} s^{-\frac{1}{2}}\, ds \;=\; 2^{-1}\alpha^{-\frac{3}{2}}.$$

Invoking (3.35) and (3.36), we then have

$$\begin{aligned}
&\int_{t_1=(1-\nu_n)/2}^{(1+\nu_n)/2} (1-t_1)^{d-3} \exp\left(-\frac{\alpha}{\bar{\varepsilon}_n t_1(1-t_1)}\right) dt_1 \\
&\sim\; 2^{-d+3}\alpha\bar{\varepsilon}_n^{\frac{1}{2}} \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n}\right) \rho_n^{\frac{1}{2}} 2^{-1}\alpha^{-\frac{3}{2}} \\
&=\; 2^{-d+3}\alpha\bar{\varepsilon}_n^{\frac{1}{2}} \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n}\right) \frac{2\alpha^{1/2}\nu_n}{\bar{\varepsilon}_n^{1/2}(1-\nu_n^2)^{1/2}} 2^{-1}\alpha^{-\frac{3}{2}} \\
&\sim\; 2^{-d+3} \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n}\right) \nu_n,
\end{aligned}$$

where "$\sim$" refers to $n \to \infty$. Plugging this into (3.34), we have

$$\begin{aligned}
p_6(n) \;\sim\; & a^2\alpha^{-\frac{d-1}{2}}\Gamma\left(\tfrac{d+1}{2}\right) \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n(1-\nu_n^2)}\right) \bar{\varepsilon}_n^{d-1} \left[\left(\tfrac{1-\nu_n}{2}\right)^{d-1} - \left(\tfrac{1+\nu_n}{2}\right)^{d-1}\right] \\
& + a^2\alpha^{-\frac{d-3}{2}}\Gamma\left(\tfrac{d+1}{2}\right) 2^{-d+3} \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n}\right) \bar{\varepsilon}_n^{d-2}\nu_n
\end{aligned}$$

as $n \to \infty$.

Recall that $T_n^{(2)}$ counts the number of exceedances of points that both fall in the narrow annulus with inner radius $1-(\varepsilon_n - \hat{\varepsilon}_n)$ and outer radius $1-\hat{\varepsilon}_n$. By plugging the asymptotic expressions of $p_5(n)$ and $p_6(n)$ into (3.33), we have

$$
\begin{aligned}
\mathbb{E}\left(T_n^{(2)}\right) &= \binom{n}{2}\mathbb{P}\left(E(1,2) \setminus B(1) \setminus B(2)\right) \\
&\leq \frac{n^2}{2}(p_4(n) + o(1)) \\
&= C_1 n^2 \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n(1-\nu_n)}\right)(\bar{\varepsilon}_n \nu_n)^{\frac{d-1}{2}} \\
&\quad + C_2 n^2 \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n(1-\nu_n^2)}\right)\bar{\varepsilon}_n^{d-1} \\
&\quad + C_3 n^2 \exp\left(-\frac{4\alpha}{\bar{\varepsilon}_n}\right)\bar{\varepsilon}_n^{d-2}\nu_n,
\end{aligned}
$$

where C_1 C_2 and C_3 are finite constants. We now study each of these summands separately by taking its logarithm and using the definitions of $\bar{\varepsilon}_n$ and ν_n. Firstly, we have

$$
\begin{aligned}
& \log C_1 + 2\log n - \frac{4\alpha}{\bar{\varepsilon}_n(1-\nu_n)} + \frac{d-1}{2}\log\bar{\varepsilon}_n + \frac{d-1}{2}\log\nu_n \\
= \;& \log C_1 + 2\log n - \frac{4\alpha}{\varepsilon_n} + 20\alpha - \frac{4\alpha\nu_n}{\varepsilon_n} + \tfrac{d-1}{2}\log\varepsilon_n + \tfrac{d-1}{2}\log\nu_n + o(1) \\
= \;& C_1 - \left(2d - \frac{5}{2}\right) l_2 n + \frac{d-1}{2} l_3 n + o(1)
\end{aligned}
$$

as $n \to \infty$, where C_1 is a finite constant and $l_3 n = \log\log\log n$. Secondly, we have

$$
\begin{aligned}
& \log C_2 + 2\log n - \frac{4\alpha}{\bar{\varepsilon}_n(1-\nu_n^2)} + (d-1)\log\bar{\varepsilon}_n \\
= \;& \log C_2 + 2\log n - \frac{4\alpha}{\varepsilon_n} + 20\alpha + (d-1)\log\varepsilon_n + o(1) \\
= \;& C_2 - \frac{1}{2} l_2 n + o(1)
\end{aligned}
$$

as n for some nite constant C_2. At last, we have

$$
\begin{aligned}
& \log C_3 + 2\log n - \frac{4\alpha}{\bar{\varepsilon}_n} + (d-2)\log\bar{\varepsilon}_n + \log\nu_n \\
= \; & \log C_3 + 2\log n - \frac{4\alpha}{\varepsilon_n} + 20\alpha + (d-2)\log\varepsilon_n + \log\nu_n + o(1) \\
= \; & C_3 - \frac{1}{2}\mathrm{l}_2 n + \mathrm{l}_3 n + o(1) \\
- \; & \quad -
\end{aligned}
$$

as n for some nite constant C_3. Consequently, we conclude that

$$\mathbb{E}\left(T_n^{(2)}\right) - \quad 0, \qquad n \qquad .$$

Finally, it follows from Markov s inequality that for each $\epsilon > 0$

$$\mathbb{P}\left(T_n^{(2)} > \epsilon\right) \leq \frac{\mathbb{E}\left(T_n^{(2)}\right)}{\epsilon} - \quad 0$$

as n , which implies the convergence of $T_n^{(2)}$ to 0 in probability. Together with $T_n^{(1)} \overset{\mathbb{P}}{-} \; 0$, we then obtain

$$T_n = T_n^{(1)} + T_n^{(2)} \overset{\mathbb{P}}{-} \; 0$$

as n .

Up to now, we have proved the following three facts about the random variable T_n:

1.) $0 < \liminf_{n} \mathbb{E}(T_n) \leq \limsup_{n} \mathbb{E}(T_n) <$,

2.) $\lim_{n} \mathbb{V}(T_n) =$,

3.) $T_n \overset{\mathbb{P}}{-} \; 0$ as n .

At rst glance these conclusions are confusing. In fact, we can explain this phenomenon by the following consideration: Only the points in the narrow annulus close to the boundary lead to the exceedances. Since the probability that a point appears in such an annulus is very small, the number of exceedances T_n converges in probability to zero. However, if a point appears in the annulus, it will lead to a cluster of exceedances, so that the expected value of T_n converges to a positive nite constant, and the variance of T_n converges to in nity.

3.6 Conical and angular supports

In this section, we study some special cases in which the support of the point distribution is a proper subset of the d-dimensional unit ball, $d \geq 2$, and the null set with respect to the probability measure $\mathbb{P}^{X_1}$ has a positive Lebesgue measure.

Let $K \subset \mathbb{B}^d$ denote the support of $\mathbb{P}^{X_1}$, and put $-K := \{x \in \mathbb{R}^d : -x \in K\}$. A necessary condition for the largest interpoint distance converging almost surely to the diameter 2 is that for each $r \in (0,1)$ the set $A_r := \{x \in \mathbb{R}^d : \|x\| \geq r\}$ satisfies $\mathbb{P}^{X_1}(A_r \setminus K) > 0$ and $\mathbb{P}^{X_1}(A_r \setminus -K) > 0$. That means, the probability for a point pair appearing in opposite directions and arbitrary close to the boundary is positive.

Firstly, we consider the case that the support of $\mathbb{P}^{X_1}$ is a set bounded by the surface of the unit ball and a circular conical surface with vertex at the origin (see Figure 3.5). Denote by K_U the support of $\mathbb{P}^{U_1}$, then K_U is the union of

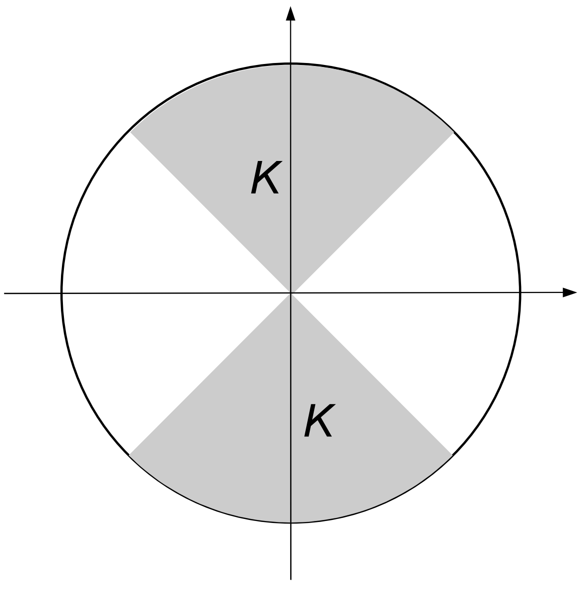

Figure 3.5: The conical support of $\mathbb{P}^{X_1}$.

the two spherical caps cut off by the conical surface. Suppose that $U_1, U_2, \ldots$ are i.i.d. with a bounded density g with respect to μ^{d-1}, then $g(u) = 0$ for all $u \in \mathbb{S}^{d-1} \setminus K_U$. Thus, the tail probability of the distance between two points has the similar lower bound $p_1(\varepsilon)$ and upper bound $p_2(\varepsilon)$ in the form of (3.8) and (3.9). The only difference is that the integral $\int_{K_U} g(u)g(-u)\mu^{d-1}(du)$ in the constant β given in (3.5) is over the support K_U of $\mathbb{P}^{U_1}$ instead of over the whole unit sphere $\mathbb{S}^{d-1}$. And consequently, the asymptotic distribution of the largest interpoint distance is also similar to the former according to the distribution of the radius.

Secondly, we consider an annular support of the point distribution. Suppose that the random points are i.i.d. in an annulus with inner radius r and outer radius 1 (see Figure 3.6). We note that the exceedance $X_1 - X_2 \geq 2 - \varepsilon$ occurs

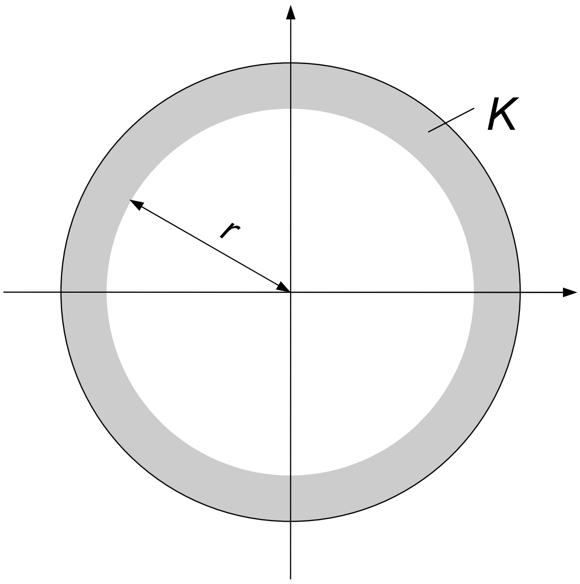

Figure 3.6: The annular support of $\mathbb{P}^{X_1}$.

only if both $X_1 \geq 1-\varepsilon$ and $X_2 \geq 1-\varepsilon$. Hence, we just need to consider the points which lie in the narrow annulus with width ε closing the boundary of the unit ball. Let $r < 1$ be fixed. Since $1-\varepsilon \leq X_1 \leq 1 \subset \leq rX_1 \leq 1$ for sufficiently small ε, the lower and the upper bound of the tail probability $\mathbb{P}(X_1 - X_2 \geq 2-\varepsilon)$ have the same asymptotic expressions (3.8) and (3.9) as ε 0. Hence, if the radial distribution of the random points belongs to the power type, the limit distribution of the largest interpoint distance is the same as in Theorem 3.6, if the radial distribution belongs to the logarithm type, the limit law is the same as in Theorem 3.11.

An extreme case of the annular support of $\mathbb{P}^{X_1}$ is the surface $\mathbb{S}^{d-1}$ of the unit ball. In this case, the radii R_i, $i = 1, 2, \ldots$, are almost surely equal to 1, and the points can be represented by the direction component, i.e. $X_i = U_i$, $i = 1, 2, \ldots$. Suppose as before that U_1 has a bounded density g with respect to μ^{d-1}. The probability for an exceedance of distance between two points is

$$
\begin{aligned}
p(\varepsilon) &:= \mathbb{P}(\|U_1 - U_2\| \geq 2 - \varepsilon) \\
&= \mathbb{P}\left(1^2 + 1^2 + 2\cos\phi \geq (2-\varepsilon)^2\right) \\
&= \mathbb{P}\left(\cos\phi \geq 1 - 2\varepsilon + \frac{1}{2}\varepsilon^2\right) \\
&= \mathbb{P}\left(\phi^2 \cos\xi \leq 4\varepsilon - \varepsilon^2\right),
\end{aligned}
$$

where $\phi = \measuredangle(U_2, -U_1)$ and $\xi \in [0, \phi]$. Recall that $1 \leq 1/\cos\xi \leq 1 + 3\varepsilon$ (see page 18), hence, $4\varepsilon - \varepsilon^2 \leq (4\varepsilon - \varepsilon^2)/\cos\xi \leq (1+3\varepsilon)(4\varepsilon - \varepsilon^2) \leq 4\varepsilon + 11\varepsilon^2$ for sufficiently small ε. By using Lemma 3.4 we obtain the asymptotic behavior of the lower and the upper bound for $p(\varepsilon)$ as follows:

$$
\begin{aligned}
p(\varepsilon) &\geq p_1(\varepsilon) := \mathbb{P}\left(\phi \leq (4\varepsilon - \varepsilon^2)^{1/2}\right) \sim \beta \cdot (4\varepsilon - \varepsilon^2)^{\frac{d-1}{2}}, \\
p(\varepsilon) &\leq p_2(\varepsilon) := \mathbb{P}\left(\phi \leq (4\varepsilon + 11\varepsilon^2)^{1/2}\right) \sim \beta \cdot (4\varepsilon + 11\varepsilon^2)^{\frac{d-1}{2}},
\end{aligned}
$$

where β is given in (3.5). Since $\lim_{\varepsilon \to 0} p_2(\varepsilon)/p_1(\varepsilon) = 1$, we have

$$
\mathbb{P}(\|X_1 - X_2\| \geq 2 - \varepsilon) \sim \beta \cdot (4\varepsilon)^{\frac{d-1}{2}}
$$

as $\varepsilon \to 0$. Set

$$
\varepsilon_n = \varepsilon_n(t) := \left(2^{d-2}\beta\right)^{-\frac{2}{d-1}} \cdot n^{-\frac{4}{d-1}} \cdot t
$$

for $t > 0$. Plugging ε_n into the tail probability, condition (3.1) reduces to

$$
\begin{aligned}
& \lim_{n\to\infty} \binom{n}{2} \mathbb{P}(\|X_1 - X_2\| > 2 - \varepsilon_n) \\
= & \lim_{n\to\infty} \frac{n^2}{2} \cdot 4^{\frac{d-1}{2}} \beta \cdot 2^{-d+2} \beta^{-1} n^{-2} t^{\frac{d-1}{2}} \\
= & \; t^{\frac{d-1}{2}} \in (0, \infty).
\end{aligned}
$$

The proof of condition (3.2) is analogous to the proof of Theorem 3.6. Thus, we have the following limit law of D_n:

Corollary 3.12. *If $X_1, X_2, \ldots$ are i.i.d. on $\mathbb{S}^{d-1}$, $d \geq 2$, with a bounded density g with respect to μ^{d-1}, we have*

$$
\lim_{n\to\infty} \mathbb{P}\left(\left(2^{d-2}\beta\right)^{\frac{2}{d-1}} \cdot n^{\frac{4}{d-1}} \cdot (2 - D_n) \leq t\right) = 1 - \exp\left(-t^{\frac{d-1}{2}}\right), \tag{3.37}
$$

where β is given in (3.5). If g is the uniform distribution on $\mathbb{S}^{d-1}$, we have $\beta = 1/\pi$.

Note that the point distribution on $\mathbb{S}^{d-1}$ can also be regarded as a special case of radial distribution of power type with $\alpha = 0$ and $a = 1$. Plugging these values into the limit law of D_n given in Theorem 3.6, we get the same formula as (3.37). Moreover, in this case the radial distribution belongs to the class of slowly varying distribution functions. Since the points are only possible to lie on the boundary of the unit ball, it is more likely to observe an exceedance of the distance over the threshold $2 - \varepsilon$. Therefore, the rescaling factor must be asymptotically greater than its counterparts in the situation of Theorem 3.11 and Theorem 3.6. Indeed, the rescaling factor of the order $O\left(n^{4/(d-1)}\right)$ is consistent with this intuitive consideration.

In Figure 3.7 there is a simulation of the limit law of the largest distance between uniformly distributed points on the unit circle.

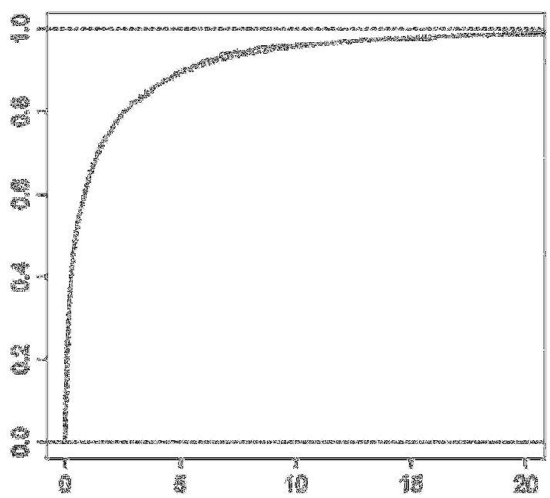

Figure 3.7: EDF of $(n^4/\pi^2) \cdot (2 - D_n)$ with $n = 1000$ random points. The dotted smooth curve is the limit law $1 - \exp\left(-\sqrt{t}\right)$.

In case $\alpha = 0$ and $a \in (0, 1)$, the points are distributed both in $\mathbb{B}^d$ and on $\mathbb{S}^{d-1}$ with

$$\mathbb{P}(\|X_1\| = 1) = \mathbb{P}(1 - \|X_1\| = 0) = F(0) = a > 0.$$

Moreover, we suppose that

$$F(s) - a \sim \tilde{a} s^{\tilde{\alpha}}$$

as $s \to 0$ for some $\tilde{a} > 0$ and $\tilde{\alpha} > 0$, which implies that the distribution function F of $1 - \|X_1\|$ jumps at 0 from 0 to a and then behaves as a power function $\tilde{a} s^{\tilde{\alpha}}$

in a small right neighborhood of 0. We can separate the tail probability of the distance between two points into the following four disjoint cases:

$$
\begin{aligned}
\mathbb{P}\Big(\|X_1 - X_2\| > 2-\varepsilon,\ X_1, X_2 \in \mathbb{S}^{d-1}\Big) &\sim C_1 \cdot \varepsilon^{\frac{d-1}{2}},\\
\mathbb{P}\Big(\|X_1 - X_2\| > 2-\varepsilon,\ X_1, X_2 \in \mathbb{B}^d \setminus \mathbb{S}^{d-1}\Big) &\sim C_2 \cdot \varepsilon^{\frac{d-1}{2}+2\tilde{\alpha}},\\
\mathbb{P}\Big(\|X_1 - X_2\| > 2-\varepsilon,\ X_1 \in \mathbb{S}^{d-1},\ X_2 \in \mathbb{B}^d \setminus \mathbb{S}^{d-1}\Big) &\sim C_3 \cdot \varepsilon^{\frac{d-1}{2}+\tilde{\alpha}},\\
\mathbb{P}\Big(\|X_1 - X_2\| > 2-\varepsilon,\ X_1 \in \mathbb{B}^d \setminus \mathbb{S}^{d-1},\ X_2 \in \mathbb{S}^{d-1}\Big) &\sim C_3 \cdot \varepsilon^{\frac{d-1}{2}+\tilde{\alpha}},
\end{aligned}
$$

where C_1, C_2, C_3 are finite positive constants. Since the first part of the tail probability dominates the others for sufficiently small ε, the limit law of D_n is the same Weibull distribution as in (3.37) with the rescaling factor of the same order $O\left(n^{\frac{4}{d-1}}\right)$, i.e.,

$$
\lim_{n\to\infty} \mathbb{P}\left(\left(2^{d-2}a^2\beta\right)^{\frac{2}{d-1}} \cdot n^{\frac{4}{d-1}} \cdot (2 - D_n) \le t\right) = 1 - \exp\left(-t^{\frac{d-1}{2}}\right).
$$

Actually, this effect of dominance by the points on $\mathbb{S}^{d-1}$ also exists, if the radial distribution function is of the logarithmic type before it jumps by a at the boundary of the unit ball.

Chapter 4

Largest area of triangles

In this chapter, we indicate how the Poisson approximation can be used in problems involving the maximum of a function of three arguments. As an example for such a kernel function, we consider the area of triangles that are formed by triplets of i.i.d. random points. Other examples could be the perimeter of triangles formed by point triplets or the average distance between three points (the maximum of such average distances was called "triameter" by Grove and Markvorsen in [19]).

To simplify the computation, we consider the case of uniformly distributed points on the unit circle. The derivation of the limit law is similar to the reasoning in Lao and Mayer [27] for the case of the largest perimeter of the triangles.

Let $X_1, X_2, \ldots$ be independent and uniformly distributed points on the unit circle $\mathbb{S}^1$, and let $A(i, j, k)$ denote the area of the triangle formed by different points X_i, X_j and X_k. The maximum area

$$A_n := \max_{1 \leq i < j < k \leq n} A(i, j, k)$$

converges almost surely to the area of an equilateral triangle with vertices on $\mathbb{S}^1$, i.e., $A_n \to 3\sqrt{3}/4$ as $n \to \infty$ with probability 1.

We first determine the asymptotic behavior of the tail probability of $A(1,2,3)$ over a threshold close to $3\sqrt{3}/4$.

Proposition 4.1. *Let X_1, X_2, X_3 be independent and uniformly distributed points on $\mathbb{S}^1$. We then have*

$$\mathbb{P}\left(A(1,2,3) \geq \frac{3\sqrt{3}}{4} - \varepsilon\right) \sim \frac{4}{3\pi}\varepsilon, \qquad \varepsilon \to 0. \tag{4.1}$$

Proof. By rotational symmetry, we may without loss of generality fix X_1 and put $X_1 = (1,0)$. Let ϕ_1, ϕ_2 be the angles (at the origin) measured counterclockwise between X_1 and X_2 and between X_1 and X_3, respectively (see Figure 4.1).

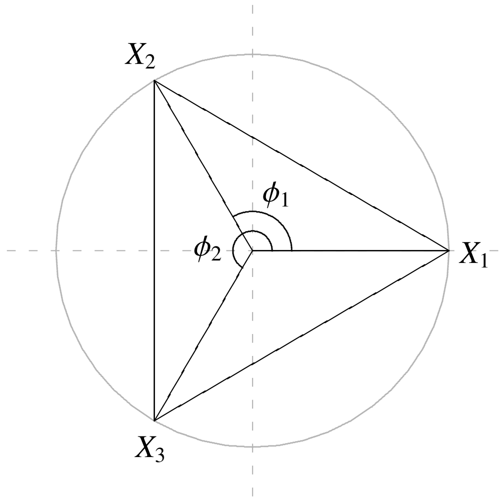

Figure 4.1: Illustration of a triangle with vertices on the unit circle.

Thus, ϕ_1 and ϕ_2 are independent and uniformly distributed in $(0, 2\pi)$. Since $\mathbb{P}(\phi_1 \le \phi_2) = 1/2$, symmetry yields

$$\mathbb{P}\left(A(1,2,3) \ge \frac{3\sqrt{3}}{4} - \varepsilon\right) = 2 \cdot \mathbb{P}\left(A(1,2,3) \ge \frac{3\sqrt{3}}{4} - \varepsilon,\ \phi_1 \le \phi_2\right).$$

Moreover, we put $\phi_1 = 2\pi/3 + \alpha_1$ and $\phi_2 = 4\pi/3 + \alpha_2$ with independent random variables α_1 uniformly distributed in $(-2\pi/3, 4\pi/3)$ and α_2 uniformly distributed in $(-4\pi/3, 2\pi/3)$. Under the condition $\phi_1 \le \phi_2$ only the point triplets with ϕ_1 close to $2\pi/3$ and ϕ_2 close to $4\pi/3$ deserve attention. We therefore consider in the following the behavior of the area for small α_1 and α_2. Suppose that $\phi_1 \le \phi_2$.

Firstly, to obtain bounds for α_2, we fix $X_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, so that $\phi_1 = 2\pi/3$. Then, $A(1,2,3) \ge \frac{3\sqrt{3}}{4} - \varepsilon$ and elementary trigonometric arguments yield

$$A(1,2,3) = \frac{\sqrt{3}}{2}\left(\cos\alpha_2 + \frac{1}{2}\right) \ge \frac{3\sqrt{3}}{4} - \varepsilon,$$

and a Taylor series expansion yields

$$|\alpha_2| \le \arccos\left(1 - 2\varepsilon/\sqrt{3}\right) = 2 \cdot 3^{-1/4}\sqrt{\varepsilon} + O\left(\varepsilon^{3/2}\right) < 2\sqrt{\varepsilon} \tag{4.2}$$

for sufficiently small $\varepsilon > 0$. Analogously, by fixing $X_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ so that $\phi_2 = 4\pi/3$, we obtain $|\alpha_1| < 2\sqrt{\varepsilon}$. Consequently, $|\alpha_2 - \alpha_1| < 4\sqrt{\varepsilon}$.

Using trigonometric formulae, the area of the triangle formed by X_1, X_2 and

X_3 is

$$\begin{aligned}
A(1,2,3) &= \frac{1}{2}\cdot\left[\sin\phi_1+\sin(\phi_2-\phi_1)+\sin(2\pi-\phi_2)\right]\\
&= \frac{1}{2}\left[\sin\left(\frac{2\pi}{3}+\alpha_1\right)-\sin\left(\frac{4\pi}{3}+\alpha_2\right)+\sin\left(\frac{2\pi}{3}+\alpha_2-\alpha_1\right)\right].
\end{aligned} \tag{4.3}$$

Taylor series expansions of α_1, α_2 and $\alpha_2-\alpha_1$ around 0 yield

$$\begin{aligned}
\sin\left(\frac{2\pi}{3}+\alpha_1\right) &= \frac{\sqrt{3}}{2}-\frac{1}{2}\alpha_1-\frac{\sqrt{3}}{4}\alpha_1^2+\frac{1}{12}\alpha_1^3+o_P\left(\alpha_1^4\right),\\
\sin\left(\frac{4\pi}{3}+\alpha_2\right) &= -\frac{\sqrt{3}}{2}-\frac{1}{2}\alpha_2+\frac{\sqrt{3}}{4}\alpha_2^2+\frac{1}{12}\alpha_2^3+o_P\left(\alpha_2^4\right),\\
\sin\left(\frac{2\pi}{3}+\alpha_2-\alpha_1\right) &= \frac{\sqrt{3}}{2}-\frac{1}{2}(\alpha_2-\alpha_1)-\frac{\sqrt{3}}{4}(\alpha_2-\alpha_1)^2\\
&\quad+\frac{1}{12}(\alpha_2-\alpha_1)^3+o_P\left((\alpha_2-\alpha_1)^4\right).
\end{aligned}$$

Putting the restrictions on α_1, α_2 and $\alpha_2-\alpha_1$ into the Taylor series expansions of the sine functions and then into (4.3), we obtain

$$\begin{aligned}
A(1,2,3) &\geq \frac{3\sqrt{3}}{4}-\frac{\sqrt{3}}{4}\left(\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2\right)-C_1\varepsilon^{3/2},\\
A(1,2,3) &\leq \frac{3\sqrt{3}}{4}-\frac{\sqrt{3}}{4}\left(\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2\right)+C_1\varepsilon^{3/2},
\end{aligned}$$

where $C_1>0$ is some constant and ε is sufficiently small. It follows that

$$\begin{aligned}
&2\mathbb{P}\left(\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2\leq\frac{4}{\sqrt{3}}\varepsilon-C_2\varepsilon^{3/2},\ \alpha_1\leq\frac{2\pi}{3}+\alpha_2\right)\\
\leq\ &\mathbb{P}\left(A(1,2,3)\geq\frac{3\sqrt{3}}{4}-\varepsilon\right)\\
\leq\ &2\mathbb{P}\left(\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2\leq\frac{4}{\sqrt{3}}\varepsilon+C_2\varepsilon^{3/2},\ \alpha_1\leq\frac{2\pi}{3}+\alpha_2\right)
\end{aligned}$$

with $C_2>0$ and ε sufficiently small. We now compute $2\mathbb{P}(\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2\leq$

$\delta,\ \alpha_1 \le \frac{2\pi}{3} + \alpha_2)$ for a sufficiently small $\delta > 0$. Conditioning on $\alpha_1 = \theta$ yields

$$
\begin{aligned}
&2\mathbb{P}\left(\alpha_1^2 + \alpha_2^2 - \alpha_1\alpha_2 \le \delta,\ \alpha_1 \le \frac{2\pi}{3} + \alpha_2\right) \\
&= \frac{1}{\pi}\int_{-2\pi/3}^{4\pi/3} \mathbb{P}\left(\theta^2 + \alpha_2^2 - \theta\alpha_2 \le \delta,\ \alpha_2 \ge \theta - \frac{2\pi}{3}\right) d\theta \\
&= \frac{1}{\pi}\int_{-2\pi/3}^{4\pi/3} \mathbb{P}\left(\left|\alpha_2 - \frac{\theta}{2}\right| \le \sqrt{\delta - \frac{3\theta^2}{4}},\ \alpha_2 \ge \theta - \frac{2\pi}{3}\right) \cdot \mathbf{1}\left\{\frac{3\theta^2}{4} \le \delta\right\} d\theta \\
&= \frac{1}{2\pi^2}\int_{-\sqrt{4\delta/3}}^{\sqrt{4\delta/3}} 2\sqrt{\delta - \frac{3\theta^2}{4}}\, d\theta \\
&= \frac{\sqrt{3}}{3\pi}\delta,
\end{aligned}
\tag{4.4}
$$

where we note that $\{|\alpha_2 - \theta/2| \le \sqrt{\delta - 3\theta^2/4}\} \subset \{\alpha_2 \ge \theta - 2\pi/3\}$ holds for sufficiently small $\delta > 0$ and $\theta \in [-\sqrt{4\delta/3}, \sqrt{4\delta/3}]$. Plugging $\delta = \frac{4}{\sqrt{3}}\varepsilon \mp C_2\varepsilon^{3/2}$ into (4.4), we obtain the following lower and upper bounds on the tail probability:

$$
\frac{\sqrt{3}}{3\pi}\left(\frac{4}{\sqrt{3}}\varepsilon - C_2\varepsilon^{3/2}\right) \le \mathbb{P}\left(A(1,2,3) \ge \frac{3\sqrt{3}}{4} - \varepsilon\right) \le \frac{\sqrt{3}}{3\pi}\left(\frac{4}{\sqrt{3}}\varepsilon + C_2\varepsilon^{3/2}\right).
$$

Consequently, (4.1) follows for $\varepsilon \to 0$. □

Figure 4.2 shows the EDF of $3\sqrt{3}/4 - A(1,2,3)$, based on a simulation with 10^5 replications. As $\varepsilon \to 0$ the EDF behaves asymptotically like the linear function $\frac{4}{3\pi}\varepsilon$ (dotted line), confirming (4.1).

In the following, we set for each $t > 0$

$$
\varepsilon_n := \varepsilon_n(t) := \frac{9\pi}{2}n^{-3}t. \tag{4.5}
$$

Hence, condition (2.4) is satisfied, i.e., we have

$$
\lim_{n\to\infty} \binom{n}{3}\mathbb{P}\left(A(1,2,3) \ge \frac{3\sqrt{3}}{4} - \varepsilon_n\right) = \lim_{n\to\infty} \frac{n^3}{6}\cdot\frac{4}{3\pi}\cdot\frac{9\pi}{2}n^{-3}t = t.
$$

However, condition (2.5) does not hold for this choice of ε_n. Recall that (2.5) is a sufficient but not necessary condition for the validity of the Poisson limit theorem given in Corollary 2.2. We shall replace this condition by the following weaker conditions:

$$
\lim_{n\to\infty} n^4\mathbb{P}\left(A(1,2,3) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ A(1,2,4) > \frac{3\sqrt{3}}{4} - \varepsilon_n\right) = 0, \tag{4.6}
$$

$$
\lim_{n\to\infty} n^5\mathbb{P}\left(A(1,2,3) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ A(1,4,5) > \frac{3\sqrt{3}}{4} - \varepsilon_n\right) = 0. \tag{4.7}
$$

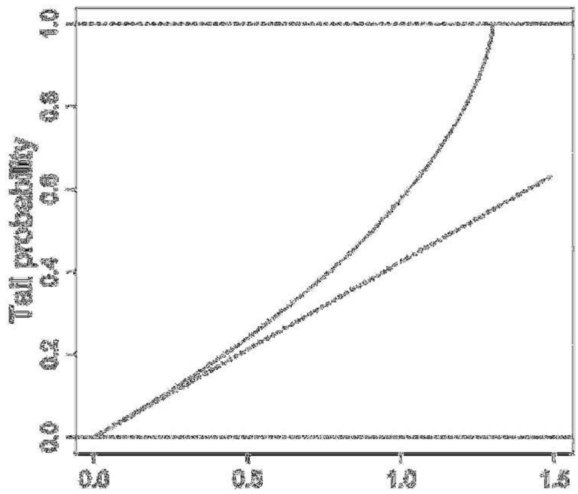

Figure 4.2: A simulation of the tail probability in (4.1) with 10^5 replications.

We now state the result of the limit law for A_n.

Theorem 4.2. *Let $X_1, X_2, \ldots$ be independent and uniformly distributed random points on $\mathbb{S}^1$. We then have for $t > 0$*

$$\lim_{n} \mathbb{P}\left(n^3 \cdot \left(\frac{3\sqrt{3}}{4} - A_n \right) \le t \right) = 1 - \exp\left(-\frac{2}{9\pi} t \right).$$

Proof. As mentioned, it remains to check conditions (4.6) and (4.7). As in Proposition 4.1, we denote by ϕ_1, ϕ_2, ϕ_3 the angles between X_1, X_2, between X_1, X_3 and between X_1, X_4, respectively, measured counterclockwise. Obviously, they are independent and uniformly distributed in $(0, 2\pi)$. There are six different orders of ϕ_1, ϕ_2 and ϕ_3, each occurring with probability $1/6$. Note that $\phi_2 \le \phi_1 \le \phi_3$ and $\phi_3 \le \phi_1 \le \phi_2$ are not possible in case of a double exceedance with overlapping points X_1, X_2. Since $\{\phi_1 \le \phi_2\} \setminus \{\phi_1 \not\le \phi_3\}$ and $\{\phi_1 \ge \phi_2\} \setminus \{\phi_1 \not\ge \phi_3\}$ are disjoint, using symmetry we have

$$\begin{aligned}
&\mathbb{P}\left(A(1,2,3) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ A(1,2,4) > \frac{3\sqrt{3}}{4} - \varepsilon_n \right) \\
=\ & 2 \cdot \mathbb{P}\left(A(1,2,3) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ A(1,2,4) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ \phi_1 \le \phi_2,\ \phi_1 \le \phi_3 \right).
\end{aligned} \tag{4.8}$$

Put $\phi_1 = \frac{2\pi}{3} + \alpha_1$, $\phi_2 = \frac{4\pi}{3} + \alpha_2$ and $\phi_3 = \frac{4\pi}{3} + \alpha_3$. Thus α_1, α_2, α_3 are independent and uniformly distributed in $(-2\pi/3, 4\pi/3)$, $(-4\pi/3, 2\pi/3)$ and $(-4\pi/3, 2\pi/3)$, respectively. By the same geometric considerations as in (4.2), we conclude that $|\alpha_i| < 2\sqrt{\varepsilon_n}$, $i = 1, 2, 3$, is a necessary condition for $A(1,2,3) > 3\sqrt{3}/4 - \varepsilon_n$ and $A(1,2,4) > 3\sqrt{3}/4 - \varepsilon_n$. Invoking (4.8) above, we then obtain

$$
\begin{aligned}
&\mathbb{P}\left(A(1,2,3) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ A(1,2,4) > \frac{3\sqrt{3}}{4} - \varepsilon_n \right) \\
\le\ & 2 \cdot \mathbb{P}\Big(\alpha_1, \alpha_2, \alpha_3 \in \big(-2\sqrt{\varepsilon_n}, 2\sqrt{\varepsilon_n}\big)\Big) \\
=\ & 2 \cdot \left(\frac{4\sqrt{\varepsilon_n}}{2\pi} \right)^3 \\
=\ & \frac{16}{\pi^3}\, \varepsilon_n^{3/2}.
\end{aligned}
$$

Plugging (4.5) into this, we therefore have

$$
\lim_{n\to\infty} n^4 \cdot \frac{16}{\pi^3} \cdot \left(\frac{9\pi}{2} n^{-3} \right)^{3/2} = 0.
$$

We now consider the two triangles formed by triplets (X_1, X_2, X_3) and (X_1, X_4, X_5) with one common vertex. Let ϕ_1, ϕ_2, ϕ_3, ϕ_4 denote the angles between X_1 and X_i, $i = 2, 3, 4, 5$, respectively, measured counterclockwise. Thus, they are independent and uniformly distributed in $(0, 2\pi)$. We consider the following four disjoint cases: $\{\phi_1 \le \phi_2\} \cap \{\phi_3 \le \phi_4\}$, $\{\phi_1 \le \phi_2\} \cap \{\phi_3 > \phi_4\}$, $\{\phi_1 > \phi_2\} \cap \{\phi_3 \le \phi_4\}$, $\{\phi_1 > \phi_2\} \cap \{\phi_3 > \phi_4\}$. By symmetry, we conclude that

$$
\begin{aligned}
&\mathbb{P}\left(A(1,2,3) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ A(1,4,5) > \frac{3\sqrt{3}}{4} - \varepsilon_n \right) \\
=\ & 4 \cdot \mathbb{P}\left(A(1,2,3) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ A(1,4,5) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ \phi_1 \le \phi_2,\ \phi_3 \le \phi_4 \right).
\end{aligned}
$$

Put $\phi_1 = \frac{2\pi}{3} + \alpha_1$, $\phi_2 = \frac{4\pi}{3} + \alpha_2$, $\phi_3 = \frac{2\pi}{3} + \alpha_3$, $\phi_4 = \frac{4\pi}{3} + \alpha_4$. Hence α_1, α_2, α_3, α_4 are independent, and α_1, α_3 are uniformly distributed in $(-2\pi/3, 4\pi/3)$, while the distribution of α_2, α_4 is uniform over $(-4\pi/3, 2\pi/3)$. By analogy with the reasoning given above, we have

$$
\begin{aligned}
& \mathbb{P}\left(A(1,2,3) > \frac{3\sqrt{3}}{4} - \varepsilon_n,\ A(1,4,5) > \frac{3\sqrt{3}}{4} - \varepsilon_n\right) \\
\leq\ & 4 \cdot \mathbb{P}\Big(\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \left(-2\sqrt{\varepsilon_n}, 2\sqrt{\varepsilon_n}\right)\Big) \\
=\ & 4 \cdot \left(\frac{4\sqrt{\varepsilon_n}}{2\pi}\right)^4 \\
=\ & \frac{64}{\pi^4}\,\varepsilon_n^2.
\end{aligned}
$$

Invoking (4.5), it follows that

$$
\lim_{n\to\infty} n^5 \cdot \frac{64}{\pi^4} \cdot \left(\frac{9\pi}{2} n^{-3} t\right)^2 = 0.
$$

Theorem 2.1 yields the result. $\square$

The simulation of the convergence given in Theorem 4.2 needs a large sample size, because the upper bound on the total variation distance in (2.2) converges to zero rather slowly. Figure 4.3 shows the simulation results of the limit law of A_n with different sample sizes, corroborating the theoretical findings.

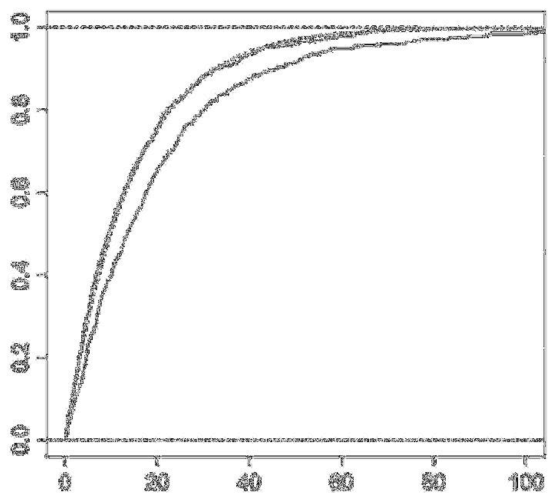

Figure 4.3: EDFs of $n^3 \cdot \left(\frac{3\sqrt{3}}{4} - A_n\right)$ with $n = 100$ (lower curve) and $n = 10000$ (upper curve), respectively. The dotted smooth curve is the limit law $1 - \exp(-\frac{2}{9\pi}t)$.

In a similar way, we can treat the largest perimeter of the triangles formed by triplets of points that are uniformly distributed on the unit circle. As in the case of area the perimeter attains its maximum value $3\sqrt{3}$, if and only if its vertices form an equilateral triangle. Denote by

$$S_n := \max_{1 \leq i < j < k \leq n} \operatorname{peri}(i, j, k)$$

the largest perimeter of the triangles formed by $X_1, \ldots, X_n$. We then have for $t > 0$

$$\lim_{n} \mathbb{P}\left(n^3 \cdot \left(3\sqrt{3} - S_n\right) \leq t\right) = 1 - e^{-\frac{2}{9\pi}t}.$$

This result was deduced by Lao and Mayer in [27].

Chapter 5

Largest distance between points on the edges of polygons

Up to now, attention has been con ned to studying the limit law of U-max-statistics by means of a Poisson limit theorem stated in Chapter 2. In some particular cases, one may derive such a limit law by classical extreme value theory and some geometric considerations. As an example we now consider the largest interpoint distance D_n between n points that are independent and uniformly distributed on the edges of a regular convex polygon in $\mathbb{R}^2$. In this case D_n converges almost surely to the length of the longest diagonals. If the polygon has an even number of sides the longest diagonals pass through the center of the polygon. Moreover, for any two diff erent diagonals the sets of endpoints are disjoint. Intuitively, only the points that are nearest to the endpoints of such a diagonal deserve attention. For large n these point sets are disjoint for diff erent diagonals, so that we can treat them separately by making use of the independence of the points. If the number of sides is odd, the longest diagonals do not meet the center of the polygon, and there are any two such diagonals starting at each vertex. Consequently, the sets of endpoints for distinct diagonals are not disjoint and we cannot use an independence property of the points.

In Section 5.1 we prove a general result on the limit law of D_n for the case that the points are distributed on a polygon with an even number of sides. In Section 5.2 we discuss the limit law of D_n in case of points on a polygon with an odd number of sides by observing an example of triangle.

5.1 Polygon with an even number of sides

Let $K \subset \mathbb{R}^2$ be the boundary of a $2m$-sided regular convex polygon, $m \in \mathbb{N}$, with vertices $v_1, \ldots, v_{2m}$. Since K exhibits rotational symmetry of order $2m$ and all its vertices lie on a common circle, we assume without loss of generality that the circumscribed circle of K is the unit circle, and that the vertices are ordered counterclockwise with $v_1 = (1, 0)$. Denote by K_i the side connecting v_i and v_{i+1}, $i = 1, \ldots, 2m$, so that $K = \bigcup_{i=1}^{2m} K_i$. Additionally, put $v_{2m+1} := v_1$ for convenience. To be more explicit, we regard K_i as a half-open interval that contains the point v_i. Split each side K_i by its midpoint into two disjoint (half-open) parts, then $K_i = K_i^{(1)} + K_i^{(2)}$, where $K_i^{(1)}$ denotes the first half from v_i to the midpoint, while $K_i^{(2)}$ denotes the second half from the midpoint to v_{i+1}.

Let $X_1, X_2, \ldots$ be independent uniformly distributed random points on K. Each of these points can be generated in two steps. We first choose one of the half-sides $K_i^{(\nu)}$, $i = 1, \ldots, 2m$ and $\nu = 1, 2$, completely at random and, independently of this choice, place the point according to a uniform distribution on $K_i^{(\nu)}$. In what follows, we identify each point on $K_i^{(1)}$ with its distance to the endpoint v_i and each point on $K_i^{(2)}$ with its distance to the endpoint v_{i+1}, $i = 1, \ldots, 2m$. Thus, the uniform distribution of the points on $K_i^{(1)}$ and on $K_i^{(2)}$ can be represented by a uniform distribution of these distances over $[0, a/2)$ and $(0, a/2]$, respectively, where a is the common length of the sides.

Let $X_1, X_2, \ldots, X_n$ be generated in that way. Writing

$$N_i^{(\nu)} := N_i^{(\nu)}(n) := \sum_{j=1}^{n} \mathbf{1}\left\{X_j \in K_i^{(\nu)}\right\}$$

for the number of points falling in $K_i^{(\nu)}$, $i = 1, \ldots, 2m$ and $\nu = 1, 2$, the random vector $\left(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)}, \ldots, N_{2m}^{(1)}, N_{2m}^{(2)}\right)$ has a multinomial distribution with parameters n and $p_i = 1/(4m)$, $i = 1, \ldots, 4m$. By the strong law of large numbers, we have

$$\frac{N_i^{(\nu)}}{n} \to \frac{1}{4m}, \qquad n \to \infty, \tag{5.1}$$

almost surely for $i = 1, \ldots, 2m$ and $\nu = 1, 2$.

Let

$$B_n := \bigcap_{i=1}^{2m} \bigcap_{\nu=1}^{2} \left\{N_i^{(\nu)} \geq 1\right\} \tag{5.2}$$

denote the event that there is at least one point on each $K_i^{(\nu)}$. We have

$$
\begin{aligned}
\mathbb{P}(B_n) &= 1-\mathbb{P}\mathop{}_{i=1}^{2m}\mathop{}_{\nu=1}^{2} N_i^{(\nu)}=0\\
&\geq 1-\mathop{}_{i=i}^{2m}\mathop{}_{\nu=1}^{2}\mathbb{P}\left(N_i^{(\nu)}=0\right)\\
&= 1-4m\cdot\ 1-\frac{1}{4m}\ ^{n}\\
&\ \ 1
\end{aligned}
$$

as n . Since for any sequence of events $(A_n)_{n\ \mathbb{N}}$ such that $\lim_n\ \mathbb{P}(A_n)$ exists we have

$$
\begin{aligned}
\lim_n \mathbb{P}(A_n) &= \lim_n \left(\mathbb{P}(A_n\ B_n)\cdot\mathbb{P}(B_n)+\mathbb{P}(A_n\ B_n^c)\cdot\mathbb{P}(B_n^c)\ \right)\\
&= \lim_n \mathbb{P}(A_n\ B_n),
\end{aligned}
$$

the limit law of D_n is the limit law of D_n conditionally on B_n. Due to this relation, the following considerations are always based on condition B_n.

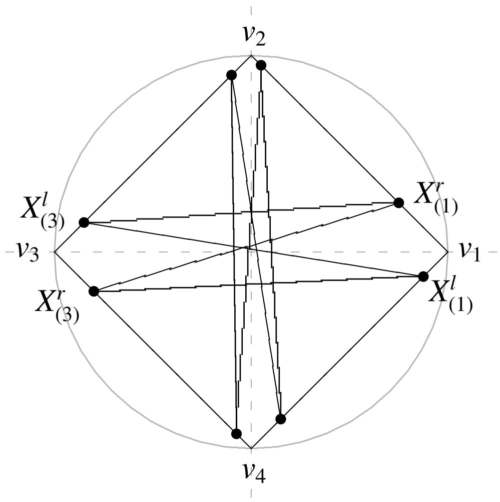

Figure 5.1: Illustration of the notations ($m = 2$).

Suppose we are standing at one of the vertices v_i and are facing the origin, then the most distant vertex from v_i is v_{m+i} for $i \leq m$ or v_{i-m} for $i > m$. Denote by $X^l_{(i)}$ and $X^r_{(i)}$ the nearest points to v_i on the left-hand side and on the right-hand side, respectively (see Figure 5.1). We then de ne for each $i = 1, \ldots, m$ a cluster of distances as follows:

$$
\begin{aligned}
D_{rl}(i, m+i) &:= \left\| X^r_{(i)} - X^l_{(m+i)} \right\| , \\
D_{rr}(i, m+i) &:= \left\| X^r_{(i)} - X^r_{(m+i)} \right\| , \\
D_{lr}(i, m+i) &:= \left\| X^l_{(i)} - X^r_{(m+i)} \right\| , \\
D_{ll}(i, m+i) &:= \left\| X^l_{(i)} - X^l_{(m+i)} \right\| .
\end{aligned}
\tag{5.3}
$$

Let X_i, X_j be any two points such that (without loss of generality) $X_i \in K_1$. Since

$$
\begin{aligned}
\left\| X_i - X_j \right\| &\leq \max\left(\left\| X_i - X^l_{(m+1)} \right\|, \left\| X_i - X^r_{(m+1)} \right\|, \left\| X_i - X^l_{(m+2)} \right\|, \left\| X_i - X^r_{(m+2)} \right\| \right) \\
&\leq \max\left(D_{rl}(1, m+1),\ D_{rr}(1, m+1),\ D_{ll}(2, m+2),\ D_{lr}(2, m+2) \right),
\end{aligned}
$$

the collection of distances given in (5.3) for $i = 1, \ldots, m$ contains all the candidates for the largest interpoint distance. Moreover, these collections are independent for different values of i, because they are formed by disjoint point sets. Let

$$D_n(i, m+i) := \max_{j,k \in \{l, r\}} D_{jk}(i, m+i)$$

denote the maximum of cluster i, $i = 1, \ldots, m$. We therefore have

$$D_n := \max_{1 \leq i < j \leq n} \left\| X_i - X_j \right\| = \max_{i=1,\ldots,m} D_n(i, m+i).$$

Since $D_n(i, m+i)$, $i = 1, \ldots, m$, are i.i.d., it is crucial to study the asymptotic behavior of $D_n(1, m+1)$. Then, the limit law of D_n follows immediately from the classical extreme value theory for i.i.d. random variables. Our result is stated in the following theorem.

Theorem 5.1. *Let $X_1, X_2, \ldots$ be independent and uniformly distributed points on the sides of a $2m$-sided regular convex polygon, $m \in \mathbb{N}$, with diameter 2. We then have for $t > 0$*

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{1}{ma} \cdot n \cdot (2 - D_n) \leq t \right) = 1 - \left(\frac{2t}{a} + 1 \right)^m \cdot \exp\left(-\frac{2m}{a} t \right),$$

where

$$a = 2 \sin \frac{\pi}{2m}$$

is the length of the sides.

Proof. We first determine some geometric quantities that are useful for later purposes. Since the regular polygon has $2m$ sides, the central angle between two neighboring vertices is $\alpha := \pi/m$, and the side length is

$$a = \sqrt{2 - 2\cos\frac{\pi}{m}} = 2\sin\frac{\pi}{2m}.$$

Each interior angle and each exterior angle is equal to $\pi - \pi/m = \pi - \alpha$ and $\pi/m = \alpha$, respectively. For each $i = 1, \dots, m$ the two sides K_i and K_{m+i} are parallel. If we extend the two sides K_i and K_{m+i-1}(or K_{m+i+1}), they will intersect at some point (see Figure 5.2). This point and the two vertices v_i, v_{m+i} constitute

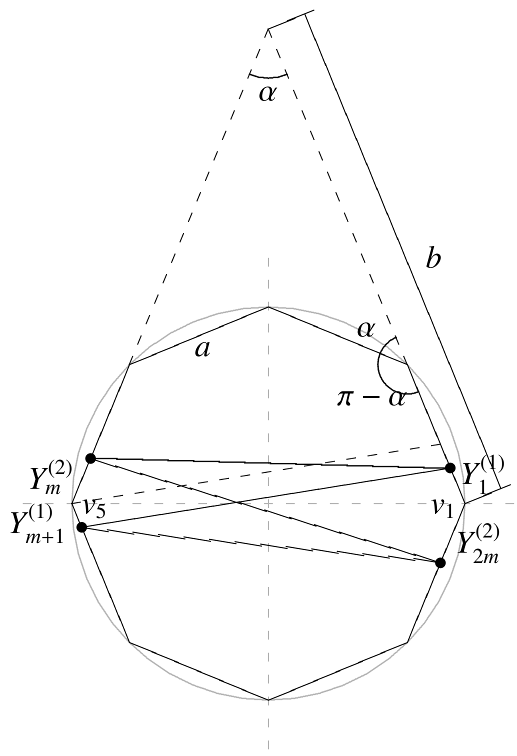

Figure 5.2: The geometric quantities ($m = 4$).

an isosceles triangle with base angles equal to half of an interior angle. Hence, the included angle between the two extension lines is equal to $\pi - (\pi - \alpha) = \alpha = \pi/m$. Denote by b the length of the distance between the point of intersection and one of the vertices v_i and v_{m+i}, that satisfies $\frac{2}{\sin(\pi/m)} = \frac{b}{\sin((\pi-\alpha)/2)}$. Plugging α into this equation, we have

$$b = \frac{2\sin\left(\frac{\pi}{2} - \frac{\pi}{2m}\right)}{\sin\frac{\pi}{m}} = \frac{2\cos\frac{\pi}{2m}}{\sin\frac{\pi}{m}} = \frac{2\cos\frac{\pi}{2m}}{2\sin\frac{\pi}{2m}\cos\frac{\pi}{2m}} = \frac{1}{\sin\frac{\pi}{2m}} = \frac{2}{a}.$$

In the following, we identify each of the critical points $X^l_{(i)}$ and $X^r_{(i)}$, $i = 1, \dots, 2m$, by its nearest distance to one of the vertices. For $i = 1, \dots, 2m$ define

$$Y_{i,N_i^{(1)}} := \begin{cases} |X^r_{(i)} - v_i| = \min_{X_j \in K_i^{(1)}} |X_j - v_i|, & \text{if } N_i^{(1)} \geq 1, \\ a/2, & \text{if } N_i^{(1)} = 0, \end{cases}$$

$$Y_{i,N_i^{(2)}} := \begin{cases} |X^r_{(i+1)} - v_{i+1}| = \min_{X_j \in K_i^{(2)}} |X_j - v_{i+1}|, & \text{if } N_i^{(2)} \geq 1, \\ a/2, & \text{if } N_i^{(2)} = 0, \end{cases}$$

where we define $X^l_{(2m+1)} := X^l_{(1)}$. Since, conditionally on B_n given in (5.2), the distances $\{|X_j - v_i| \mid X_j \in K_i^{(1)}\}$, $\{|X_j - v_{i+1}| \mid X_j \in K_i^{(2)}\}$, $i = 1, 2, \dots, 2m$, are independent and $\mathcal{U}(0, a/2)$-distributed, the random variables $Y_{i,N_i^{(1)}}$, $Y_{i,N_i^{(2)}}$ ($i = 1, \dots, 2m$) are extremes of i.i.d. random variables with random sample sizes $N_i^{(1)}$ and $N_i^{(2)}$, respectively. The classical extreme value theory yields $\mathbb{P}\left(n \cdot \min_{1 \leq j \leq n} Z_j \leq t\right) = 1 - \exp(-2t/a)$, if the random variables $(Z_j)_{j \in \mathbb{N}}$ are independent and $\mathcal{U}(0, a/2)$-distributed. By a similar method used in the proof of Lemma 6.2.3 in Galambos [17], we obtain for $i = 1, \dots, 2m$ and $\nu = 1, 2$

$$N_i^{(\nu)} \cdot Y_{i,N_i^{(\nu)}} \xrightarrow{\mathcal{D}} E_i^{(\nu)}$$

as $n \to \infty$, where $\left(E_i^{(\nu)} : i = 1, \dots, 2m; \nu = 1, 2\right)$ are independent and exponential distributed random variables with parameter $2/a$. More precisely, $E_i^{(\nu)}$ has density $f(t) = \frac{2}{a}\exp(-\frac{2t}{a})$ and distribution function $F(t) = 1 - \exp(-\frac{2t}{a})$ for $t \geq 0$. Invoking (5.1), we have

$$\frac{n}{4m} \cdot Y_{i,N_i^{(\nu)}} = \frac{1/4m}{N_i^{(\nu)}/n} \cdot N_i^{(\nu)} Y_{i,N_i^{(\nu)}} \xrightarrow{\mathcal{D}} E_i^{(\nu)} \tag{5.4}$$

as $n \to \infty$.

As hinted, the pivot of the proof is to derive the limit law of $D_n(1, m+1)$. We first represent the four distances $D_{rl}(1, m+1)$, $D_{rr}(1, m+1)$, $D_{lr}(1, m+1)$ and $D_{ll}(1, m+1)$ given in (5.3) with the help of $Y_{i,N_i^{(\nu)}}$, $i = 1, \dots, 2m$ and $\nu = 1, 2$. Suppose that condition B_n holds. To simplify notations, we will write $Y_i^{(\nu)} := Y_{i,N_i^{(\nu)}}$ in the following.

Using the law of cosines and plugging $\alpha = \pi/m$ and $b = (\sin \frac{\pi}{2m})^{-1} = 2/a$

into the expression, we get

$$
\begin{aligned}
& D_{rl}(1, m+1) \\
= & \sqrt{\left(b - Y_1^{(1)}\right)^2 + \left(b - Y_m^{(2)}\right)^2 - 2\left(b - Y_1^{(1)}\right)\left(b - Y_m^{(2)}\right)\cos\alpha} \\
= & \sqrt{2b(1-\cos\alpha)\left(b - Y_1^{(1)} - Y_m^{(2)}\right) + Y_1^{(1)2} + Y_m^{(2)2} - 2Y_1^{(1)}Y_m^{(2)}\cos\alpha} \\
= & \sqrt{2a\left(\frac{2}{a} - Y_1^{(1)} - Y_m^{(2)}\right) + Y_1^{(1)2} + Y_m^{(2)2} - 2Y_1^{(1)}Y_m^{(2)}\cos\frac{\pi}{m}}.
\end{aligned}
$$

Since $-1 \leq \cos\frac{\pi}{m} \leq 1$, we have on one hand

$$
\begin{aligned}
D_{rl}(1, m+1) \ \geq & \ \sqrt{2a\left(\frac{2}{a} - Y_1^{(1)} - Y_m^{(2)}\right) + \left(Y_1^{(1)} - Y_m^{(2)}\right)^2} \\
\geq & \ \sqrt{2a\left(\frac{2}{a} - Y_1^{(1)} - Y_m^{(2)}\right)}
\end{aligned}
$$

and on the other hand

$$
D_{rl}(1, m+1) \ \leq \ \sqrt{2a\left(\frac{2}{a} - Y_1^{(1)} - Y_m^{(2)}\right) + \left(Y_1^{(1)} + Y_m^{(2)}\right)^2}.
$$

Obviously, both $Y_1^{(1)}$ and $Y_m^{(2)}$ converge in probability to 0 as $n \to \infty$. Invoking (5.4) we have

$$
\frac{n}{4m} \cdot \left(Y_1^{(1)} + Y_m^{(2)}\right) \xrightarrow{\mathcal{D}} E_1^{(1)} + E_m^{(2)}, \qquad n \to \infty.
$$

Since $Y_1^{(1)} + Y_m^{(2)} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, it follows from Sluzky's lemma that

$$
\frac{n}{4m} \cdot \left(Y_1^{(1)} + Y_m^{(2)}\right)^k = \frac{n}{4m} \cdot \left(Y_1^{(1)} + Y_m^{(2)}\right) \cdot \left(Y_1^{(1)} + Y_m^{(2)}\right)^{k-1} \xrightarrow{\mathbb{P}} 0
$$

as $n \to \infty$ for each $k > 1$.

Regarding the lower and the upper bound of $D_{rl}(1, m+1)$ as functions of $Y_1^{(1)} + Y_m^{(2)}$ and taking their Taylor series expansions around 0, we find that the first order Taylor approximations coincide. Hence, we obtain

$$
D_{rl}(1, m+1) = 2 - \frac{a}{2}\left(Y_1^{(1)} + Y_m^{(2)}\right) + o_P\left(Y_1^{(1)} + Y_m^{(2)}\right),
$$

and Sluzky s lemma yields

$$\frac{n}{2ma} \cdot (2 - D_{rl}(1, m+1)) \xrightarrow{\mathcal{D}} E_1^{(1)} + E_m^{(2)}$$

as n .

To derive the limit law of $D_{rr}(1, m+1)$ we draw an auxiliary line that passes through the vertex v_{m+1} and is parallel to the line formed by $X^r_{(1)}$ and $X^r_{(m+1)}$ (see Figure 5.2). It follows from the law of cosines that

$$\begin{aligned} & D_{rr}(1, m+1) \\ = \quad & \sqrt{b^2 + \left(b - Y_1^{(1)} - Y_{m+1}^{(1)}\right)^2 - 2b\left(b - Y_1^{(1)} - Y_{m+1}^{(1)}\right)\cos\alpha} \\ = \quad & \sqrt{2b(1-\cos\alpha)\left(b - Y_1^{(1)} - Y_{m+1}^{(1)}\right) + \left(Y_1^{(1)} + Y_{m+1}^{(1)}\right)^2} \\ = \quad & \sqrt{2a\ \frac{2}{a} - Y_1^{(1)} - Y_{m+1}^{(1)}\ + \left(Y_1^{(1)} + Y_{m+1}^{(1)}\right)^2}. \end{aligned}$$

Arguing as above, we have

$$D_{rr}(1, m+1) = 2 - \frac{a}{2}\left(Y_1^{(1)} + Y_{m+1}^{(1)}\right) + o_P\left(Y_1^{(1)} + Y_{m+1}^{(1)}\right).$$

Analogously to $D_{rl}(1, m+1)$ and $D_{rr}(1, m+1)$, we obtain

$$\begin{aligned} D_{lr}(1, m+1) &= 2 - \frac{a}{2}\left(Y_{m+1}^{(1)} + Y_{2m}^{(2)}\right) + o_P\left(Y_{m+1}^{(1)} + Y_{2m}^{(2)}\right), \\ D_{ll}(1, m+1) &= 2 - \frac{a}{2}\left(Y_m^{(2)} + Y_{2m}^{(2)}\right) + o_P\left(Y_m^{(2)} + Y_{2m}^{(2)}\right). \end{aligned}$$

Note that the rst order Taylor polynomials of the four distances coincide. Since $D_n(1, m+1) = \max_{j,k\ \ ,l\ r}\{D_{jk}(1, m+1)\}$, we conclude that

$$\begin{aligned} & \frac{n}{2ma} \cdot (2 - D_n(1, m+1)) \\ = \quad & \min_{j,k\ \ ,l\ r} \frac{n}{2ma} \cdot \{2 - D_{jk}(1, m+1)\} \\ = \quad & \min\left\{\frac{n}{4m}\left(Y_1^{(1)} + Y_m^{(2)}\right),\ \frac{n}{4m}\left(Y_1^{(1)} + Y_{m+1}^{(1)}\right)\right. \\ & \left.\frac{n}{4m}\left(Y_{m+1}^{(1)} + Y_{2m}^{(2)}\right),\ \frac{n}{4m}\left(Y_m^{(2)} + Y_{2m}^{(2)}\right)\right\} + o_P(1). \end{aligned} \tag{5.5}$$

From (5.4) and the independence of $Y_i^{(\nu)}$, $i = 1, \ldots, 2m$ and $\nu = 1, 2$, it is obvious that

$$\frac{n}{4m} \cdot \left(Y_1^{(1)}, Y_m^{(2)}, Y_{m+1}^{(1)}, Y_{2m}^{(2)}\right) \xrightarrow{\mathcal{D}} \left(E_1^{(1)}, E_m^{(2)}, E_{m+1}^{(1)}, E_{2m}^{(2)}\right)$$

as $n \to \infty$. Let $h : \mathbb{R}^4 \to \mathbb{R}$ be defined by

$$h(y_1, y_2, y_3, y_4) := \min\left(y_1 + y_2,\ y_1 + y_3,\ y_3 + y_4,\ y_2 + y_4\right).$$

The continuous mapping theorem and (5.5) then yield

$$\begin{aligned} & \frac{n}{2ma} \cdot (2 - D_n(1, m+1)) \\ \xrightarrow{\mathcal{D}} \quad & \min\left(E_1^{(1)} + E_m^{(2)},\ E_1^{(1)} + E_{m+1}^{(1)},\ E_{m+1}^{(1)} + E_{2m}^{(2)},\ E_m^{(2)} + E_{2m}^{(2)}\right) \\ =: \quad & E \end{aligned}$$

as $n \to \infty$.

Since $E_1^{(1)}$, $E_m^{(2)}$, $E_{m+1}^{(1)}$ and $E_{2m}^{(2)}$ are independent and $Exp(2/a)$-distributed, we can deduce the distribution function of E conveniently. The details can be found in Appendix D.1. The result is

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{2ma} \cdot n \cdot (2 - D_n(1, m+1)) \le t\right) = 1 - \left(\frac{4t}{a} + 1\right) \cdot \exp\left(-\frac{4t}{a}\right)$$

for each $t \ge 0$. Applying the independence of $D_n(i, m+i)$, $i = 1, \dots, m$, and the relation $D_n = \max_{i=1,\dots,m} D_n(i, m+i)$, we have

$$\begin{aligned} & \lim_{n\to\infty} \mathbb{P}\left(\frac{1}{2ma} \cdot n \cdot (2 - D_n) \le t\right) \\ = \quad & \lim_{n\to\infty} \mathbb{P}\left(\min_{i=1,\dots,m}\left(\frac{1}{2ma} \cdot n \cdot (2 - D_n(i, m+i))\right) \le t\right) \\ = \quad & 1 - \left(\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{2ma} \cdot n \cdot (2 - D_n(1, m+1)) > t\right)\right)^m \\ = \quad & 1 - \left(\frac{4t}{a} + 1\right)^m \cdot \exp\left(-\frac{4m}{a} t\right). \end{aligned}$$

By a simple transformation, we get the limit law stated in Theorem 5.1. □

As examples, we consider the cases $m = 1$ and $m = 2$.

If $m = 1$, the random points are uniformly distributed on a diameter of the unit circle and $a = 2 \sin\frac{\pi}{2} = 2$. The limit law of D_n is then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{n}{2} \cdot (2 - D_n) \le t\right) = 1 - (t+1)\, e^{-t},$$

which coincides with the limit law given in (1.1).

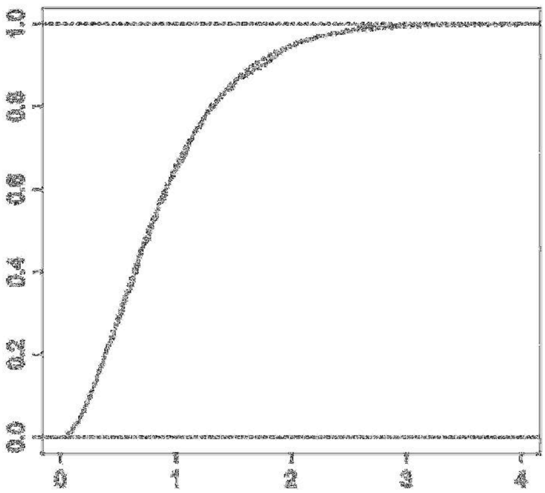

Figure 5.3: The simulation of the limit law of D_n, when the points are distributed on the sides of a square.

If $m = 2$, the random points are uniformly distributed on the sides of a square with side length $a = 2\sin\frac{\pi}{4} = \sqrt{2}$. Thus, the limit law of D_n reduces to

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\sqrt{2}}{4}\, n\,(2 - D_n) \le t\right) = 1 - (\sqrt{2}\,t + 1)^2 \cdot e^{-2\sqrt{2}\,t} \tag{5.6}$$

for $t > 0$. Figure 5.3 gives a concrete simulation of this limit law. Firstly, we generated 500 points that are independent and uniformly distributed on the sides of a square with side length $\sqrt{2}$, and then determined their largest interpoint distance. This procedure was repeated 500 times. Figure 5.3 shows the EDF of $\frac{\sqrt{2}}{4} \cdot 500 \cdot (2 - D_{500})$. The dotted smooth curve in Figure 5.3 is the distribution function figuring on the right-hand side of (5.6). The EDF approximates the smooth curve very well, which corroborates our theoretical findings.

5.2 Polygon with an odd number of sides

Let $X_1, X_2, \ldots$ be independent uniformly distributed random points on the boundary $K \subset \mathbb{R}^2$ of a $(2m+1)$-sided regular convex polygon, $m \in \mathbb{N}$. As before, we denote the vertices counterclockwise by $v_1, \ldots, v_{2m+1}$ with $v_1 = (1, 0)$ and denote by K_i the side connecting v_i and v_{i+1}, $i = 1, \ldots, 2m+1$, so that $K = \bigcup_{i=1}^{2m+1} K_i$. Additionally, put $v_{2m+2} := v_1$. We split each side K_i by its midpoint into two disjoint half-open parts $K_i^{(1)}$ and $K_i^{(2)}$.

Let

$$N_i^{(\nu)} := N_i^{(\nu)}(n) := \sum_{j=1}^{n} \mathbf{1}\left(X_j \in K_i^{(\nu)}\right)$$

be the number of points in $K_i^{(\nu)}$, $i = 1, \ldots, 2m+1$ and $\nu = 1, 2$. By the strong law of large numbers, we have

$$\frac{N_i^{(\nu)}}{n} \longrightarrow \frac{1}{2(2m+1)}, \qquad n \to \infty,$$

almost surely for $i = 1, \ldots, 2m+1$ and $\nu = 1, 2$. The following considerations are always based on the condition

$$B_n := \bigcap_{i=1}^{2m+1} \bigcap_{\nu=1}^{2} \left\{ N_i^{(\nu)} \geq 1 \right\}$$

which states that there is at least one point on each half-side $K_i^{(\nu)}$. Note that $\lim_{n \to \infty} \mathbb{P}(B_n) = 1$.

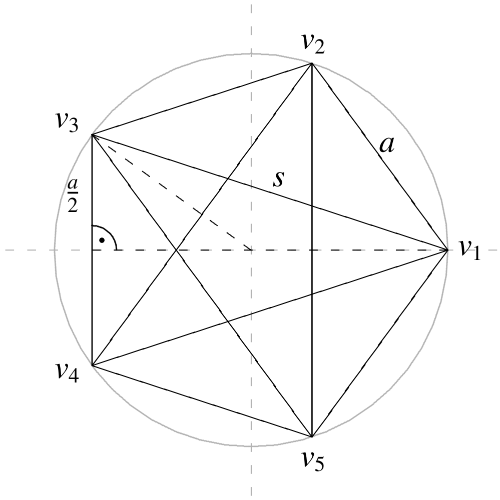

Figure 5.4: Illustration of the geometric quantities in a pentagon ($m = 2$).

In a regular convex polygon with $2m + 1$ sides, the central angle between two neighboring vertices is $\alpha := 2\pi/(2m + 1)$, and the side length is

$$a = \sqrt{2 - 2\cos\frac{2\pi}{2m+1}} = 2\sin\frac{\pi}{2m+1}.$$

The following schema gives all the possible pairs of endpoints, which form the longest diagonals:

$$v_1 \begin{smallmatrix} \nearrow v_{m+1} \\ \searrow v_{m+2} \end{smallmatrix} \quad v_2 \begin{smallmatrix} \nearrow v_{m+2} \\ \searrow v_{m+3} \end{smallmatrix} \quad \ldots \quad v_i \begin{smallmatrix} \nearrow v_{m+i} \\ \searrow v_{m+i+1} \end{smallmatrix} \quad \ldots \quad v_m \begin{smallmatrix} \nearrow v_{2m} \\ \searrow v_{2m+1} \end{smallmatrix} \quad v_{m+1} \begin{smallmatrix} \nearrow v_{2m+1} \\ \searrow v_{2m+2} = v_1 \end{smallmatrix}$$

where $\diagup$ and $\diagdown$ mean the connections by one of the longest diagonals. Thus, there are altogether $2m + 1$ diagonals with the maximal length

$$s = \quad 1 + \quad \overline{1 - \frac{a^2}{4}}^{\,2} + \frac{a^2}{4} \quad {}^{1/2} = 2 + \quad \overline{4 - a^2}^{\;1/2} .$$

We adopt the notations $X^l_{(i)}$, $X^r_{(i)}$ for the nearest points to v_i on the left- and right-hand side. Then, for each $i = 1, \ldots, m + 1$ there is a cluster of distances de ned by

$$\begin{aligned} D_{rl}(i, m+i) &:= \quad X^r_{(i)} - X^l_{(m+i)} \quad , \\ D_{rr}(i, m+i) &:= \quad X^r_{(i)} - X^r_{(m+i)} \quad , \\ D_{lr}(i, m+i) &:= \quad X^l_{(i)} - X^r_{(m+i)} \quad , \\ D_{ll}(i, m+i) &:= \quad X^l_{(i)} - X^l_{(m+i)} \quad , \end{aligned} \tag{5.7}$$

and for each $i = 1, \ldots, m$ there is a cluster of distances de ned by

$$\begin{aligned} D_{rl}(i, m+i+1) &:= \quad X^r_{(i)} - X^l_{(m+i+1)} \quad , \\ D_{rr}(i, m+i+1) &:= \quad X^r_{(i)} - X^r_{(m+i+1)} \quad , \\ D_{lr}(i, m+i+1) &:= \quad X^l_{(i)} - X^r_{(m+i+1)} \quad , \\ D_{ll}(i, m+i+1) &:= \quad X^l_{(i)} - X^l_{(m+i+1)} \quad . \end{aligned} \tag{5.8}$$

Since for any distinct points X_i, X_j such that (without loss of generality) $X_i \quad K_1$ the following inequality holds:

$$\begin{aligned} & X_i - X_j \\ \le \;& \max \quad X_i - X^l_{(m+1)} \quad , X_i - X^l_{(m+2)} \quad , X_i - X^l_{(m+3)} \quad , \\ & \qquad X_i - X^r_{(m+1)} \quad , X_i - X^r_{(m+2)} \quad , X_i - X^r_{(m+3)} \quad , \\ \le \;& \max \; D_{rl}(1, m+1),\; D_{rl}(1, m+2),\; D_{ll}(2, m+2),\; D_{ll}(2, m+3), \\ & \qquad D_{rr}(1, m+1),\; D_{rr}(1, m+2),\; D_{lr}(2, m+2),\; D_{lr}(2, m+3) \;, \end{aligned}$$

the collection of the distances given in (5.7) and (5.8) contains all the $4 \cdot (2m+1)$ candidates for the largest interpoint distance. De ne for each $i = 1, \ldots, m + 1$

$$D_n(i, m+i) \;:=\; \max_{j,k \quad ,l \; r} D_{jk}(i, m+i)$$

and for each $i = 1, \ldots, m$

$$D_n(i, m+i+1) \;:=\; \max_{j,k \quad ,l\, r} D_{jk}(i, m+i+1).$$

Notice that these random variables are not independent, so that the computation cannot be simplified as in Section 5.1. In what follows, we derive the limit law of D_n, when the points are on the sides of an equilateral triangle.

Theorem 5.2. *Let $X_1, X_2, \ldots$ be independent and uniformly distributed points on the sides of a equilateral triangle with radius* 1 *(from the centre to one of the vertices). We then have for $t > 0$*

$$\lim_{n} \mathbb{P}\left(\frac{n}{3} \cdot \left(\sqrt{3} - D_n\right) \le t\right)$$
$$= 1 - \exp\left(-2\sqrt{3}t \cdot \left(\frac{3}{2} + 2\sqrt{3}t + \frac{569}{384}t^2 + \frac{1043}{9216}\sqrt{3}t^3\right) + \frac{1}{2} \cdot \exp\left(-\frac{25}{12}\sqrt{3}t\right) - \exp\left(-\frac{53}{24}\sqrt{3}t\right) + \exp\left(-\frac{13}{6}\sqrt{3}t\right)\right). \tag{5.9}$$

Proof. In the case $m = 1$ we have a equilateral triangle with side length $a = \sqrt{3}$ and the maximal diagonal length $s = \sqrt{3}$. There are 12 candidates for the largest interpoint distance, which are plotted in Figure 5.5.

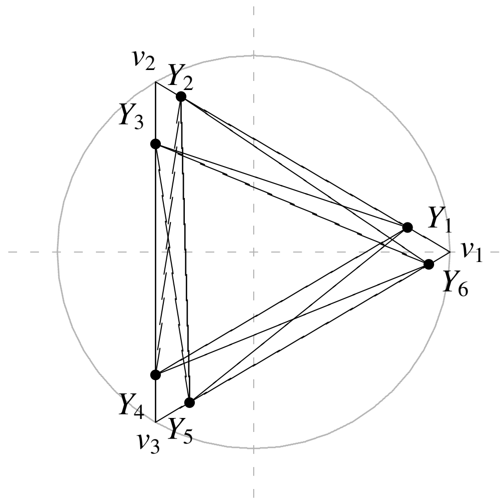

Figure 5.5: Candidates for the largest interpoint distance in the case $m = 1$.

Suppose that condition B_n holds. We now identify the endpoints of these candidates by its distance to the nearest vertex. Define

$$
\begin{aligned}
Y_1 &:= Y_{1,N_1^{(1)}} := \left|X^r_{(1)} - v_1\right| = \min_{X_j \in K_1^{(1)}} \left|X_j - v_1\right| ,\\
Y_2 &:= Y_{1,N_1^{(2)}} := \left|X^l_{(2)} - v_2\right| = \min_{X_j \in K_1^{(2)}} \left|X_j - v_2\right| ,\\
Y_3 &:= Y_{2,N_2^{(1)}} := \left|X^r_{(2)} - v_2\right| = \min_{X_j \in K_2^{(1)}} \left|X_j - v_2\right| ,\\
Y_4 &:= Y_{2,N_2^{(2)}} := \left|X^l_{(3)} - v_3\right| = \min_{X_j \in K_2^{(2)}} \left|X_j - v_3\right| ,\\
Y_5 &:= Y_{3,N_3^{(1)}} := \left|X^r_{(3)} - v_3\right| = \min_{X_j \in K_3^{(1)}} \left|X_j - v_3\right| ,\\
Y_6 &:= Y_{3,N_3^{(2)}} := \left|X^l_{(1)} - v_1\right| = \min_{X_j \in K_3^{(2)}} \left|X_j - v_1\right| .
\end{aligned}
$$

By the same considerations as in Section 5.1 and $N_i^{(\nu)}/n \longrightarrow 1/6$ as $n \to \infty$, we conclude that

$$
\frac{n}{6}\cdot Y_1 = \frac{1/6}{N_1^{(1)}/n}\cdot N_1^{(1)} Y_1 \xrightarrow{\mathcal{D}} E_1 \tag{5.10}
$$

as $n \to \infty$, where E_1 is an exponential distributed random variable with parameter $2/\sqrt{3}$. Moreover, this convergence holds for each Y_i, $i = 1, \ldots, 6$, when rescaled by $n/6$, with i.i.d. limit laws $E_1, \ldots, E_6$, respectively.

Now, all the 12 candidates can be represented with the help of $Y_1, \ldots, Y_6$. Since under condition B_n the random variables $Y_1, \ldots, Y_6$ are independent, we can obtain the limit law of D_n by using the continuous mapping theorem. We first treat four candidates in some detail. To begin with, we have

$$
D_{rl}(1,2) = \sqrt{3} - Y_1 - Y_2 = \sqrt{3} - (Y_1 + Y_2)\,.
$$

By using the law of cosines we have

$$
\begin{aligned}
D_{rr}(1,2) &= \sqrt{(\sqrt{3} - Y_1)^2 + Y_3^2 - (\sqrt{3} - Y_1)Y_3}\\
&= \sqrt{3 - 2\sqrt{3}Y_1 - \sqrt{3}Y_3 + Y_1^2 + Y_3^2 + Y_1Y_3}\\
&\geq \sqrt{3 - \left(2\sqrt{3}Y_1 + \sqrt{3}Y_3\right)}
\end{aligned}
$$

and on the other hand

$$
D_{rr}(1,2) \leq \sqrt{3 - \left(2\sqrt{3}Y_1 + \sqrt{3}Y_3\right) + \left(2\sqrt{3}Y_1 + \sqrt{3}Y_3\right)^2}
$$

Since these bounds are both functions of $2\sqrt{3}Y_1 + \sqrt{3}Y_3$, we take first order Taylor approximations and obtain

$$D_{rr}(1,2) = \sqrt{3} - \left(Y_1 + \frac{1}{2}Y_3\right) + o_P\Big(2\sqrt{3}Y_1 + \sqrt{3}Y_3\Big).$$

Using similar considerations, we have

$$\begin{aligned}
D_{rl}(1,3) &= \sqrt{(\sqrt{3}-Y_1)^2 + (\sqrt{3}-Y_4)^2 - (\sqrt{3}-Y_1)(\sqrt{3}-Y_4)} \\
&= \sqrt{3 - \sqrt{3}Y_1 - \sqrt{3}Y_4 + Y_1^2 + Y_4^2 - Y_1Y_4} \\
&= \sqrt{3} - \left(\frac{1}{2}Y_1 + \frac{1}{2}Y_4\right) + o_P\Big(\sqrt{3}Y_1 + \sqrt{3}Y_4\Big)
\end{aligned}$$

and

$$\begin{aligned}
D_{rr}(1,3) &= \sqrt{Y_1^2 + (\sqrt{3}-Y_5)^2 - Y_1(\sqrt{3}-Y_5)} \\
&= \sqrt{3 - \sqrt{3}Y_1 - 2\sqrt{3}Y_5 + Y_1^2 + Y_5^2 + Y_1Y_5} \\
&= \sqrt{3} - \left(\frac{1}{2}Y_1 + Y_5\right) + o_P\Big(\sqrt{3}Y_1 + 2\sqrt{3}Y_5\Big).
\end{aligned}$$

The other distances can be obtained similarly. We have

$$\begin{aligned}
D_{ll}(2,3) &= \sqrt{3} - \left(\frac{1}{2}Y_2 + Y_4\right) + o_P\Big(\sqrt{3}Y_2 + 2\sqrt{3}Y_4\Big), \\
D_{lr}(2,3) &= \sqrt{3} - \left(\frac{1}{2}Y_2 + \frac{1}{2}Y_5\right) + o_P\Big(\sqrt{3}Y_2 + \sqrt{3}Y_5\Big), \\
D_{ll}(1,2) &= \sqrt{3} - \left(Y_2 + \frac{1}{2}Y_6\right) + o_P\Big(2\sqrt{3}Y_2 + \sqrt{3}Y_6\Big), \\
D_{rl}(2,3) &= \sqrt{3} - (Y_3 + Y_4)\,, \\
D_{rr}(2,3) &= \sqrt{3} - \left(Y_3 + \frac{1}{2}Y_5\right) + o_P\Big(2\sqrt{3}Y_3 + \sqrt{3}Y_5\Big), \\
D_{lr}(1,2) &= \sqrt{3} - \left(\frac{1}{2}Y_3 + \frac{1}{2}Y_6\right) + o_P\Big(\sqrt{3}Y_3 + \sqrt{3}Y_6\Big), \\
D_{ll}(1,3) &= \sqrt{3} - \left(\frac{1}{2}Y_4 + Y_6\right) + o_P\Big(\sqrt{3}Y_4 + 2\sqrt{3}Y_6\Big), \\
D_{lr}(1,3) &= \sqrt{3} - (Y_5 + Y_6)\,.
\end{aligned}$$

Since

$$\begin{aligned} D_n &= \max\Big\{ D_{rl}(1,2),\ D_{rr}(1,2),\ D_{rl}(1,3),\ D_{rr}(1,3),\ D_{ll}(2,3),\ D_{lr}(2,3), \\ &\qquad\quad D_{ll}(1,2),\ D_{rl}(2,3),\ D_{rr}(2,3),\ D_{lr}(1,2),\ D_{ll}(1,3),\ D_{lr}(1,3)\Big\}, \end{aligned}$$

we apply (5.10) and the continuous mapping theorem and obtain the limit law as $n \to \infty$ as follows:

$$\begin{aligned} &\frac{n}{3}\cdot\left(\sqrt{3}-D_n\right) \\ =\ & 2\cdot\frac{n}{6}\cdot\min\Big\{ \sqrt{3}-D_{rl}(1,2),\quad \sqrt{3}-D_{rr}(1,2),\quad \sqrt{3}-D_{rl}(1,3), \\ &\qquad\qquad \sqrt{3}-D_{rr}(1,3),\quad \sqrt{3}-D_{ll}(2,3),\quad \sqrt{3}-D_{lr}(2,3), \\ &\qquad\qquad \sqrt{3}-D_{ll}(1,2),\quad \sqrt{3}-D_{rl}(2,3),\quad \sqrt{3}-D_{rr}(2,3), \\ &\qquad\qquad \sqrt{3}-D_{lr}(1,2),\quad \sqrt{3}-D_{ll}(1,3),\quad \sqrt{3}-D_{lr}(1,3)\Big\} \\ \xrightarrow{\mathcal{D}}\ & 2\cdot\min\Big\{E_1+E_2,\ E_1+\frac{1}{2}E_3,\ \frac{1}{2}E_1+\frac{1}{2}E_4,\ \frac{1}{2}E_1+E_5, \\ &\qquad \frac{1}{2}E_2+E_4,\ \frac{1}{2}E_2+\frac{1}{2}E_5,\ E_2+\frac{1}{2}E_6,\ E_3+E_4, \\ &\qquad E_3+\frac{1}{2}E_5,\ \frac{1}{2}E_3+\frac{1}{2}E_6,\ \frac{1}{2}E_4+E_6,\ E_5+E_6\Big\} \\ =:\ & 2\cdot E. \end{aligned} \tag{5.11}$$

The distribution function of E is obtained in Appendix D.2 by long and tedious computations. Then, the limit law can be obtained by a simple transformation. □

Figure 5.6 shows the EDF of $\frac{500}{6}\cdot\left(\sqrt{3}-D_{500}\right)$, based on 500 replications. The dotted smooth curve shows the limit distribution function given on the right-hand side of (5.9).

In principle the method in the proof of Theorem 5.2 can be generalized to a general regular convex polygon with an odd number of sides. However, the analytical complexity grows rapidly with the number of sides, since we cannot exploit an independence property that holds for polygons with an even number of sides.

Consider a regular convex polygon with $2m+1$ sides, $m \geq 2$. The interior angles and the exterior angles are equal to $\pi-\alpha = (2m-1)\pi/(2m+1)$ and $\alpha = 2\pi/(2m+1)$, respectively. For further purposes, it requires knowing more

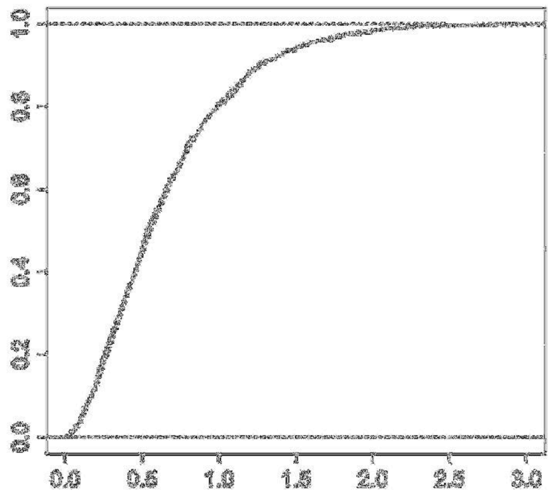

Figure 5.6: EDF of $\frac{500}{6} \cdot \left(\sqrt{3} - D_{500}\right)$ and limit distribution function (5.9) in case of an equilateral triangle.

geometric quantities with respect to each longest diagonal. By the symmetry of the polygon, we consider for instance the longest diagonal connection v_2 and v_{m+2} (see Figure 5.7). The extension lines of the two sides K_2 and K_{m+1} intersect at a point u, the origin and points u, v_2, v_{m+2} shape a kite. Since the central angle between v_2 and v_{m+1} is $2m\pi/(2m+1)$ and the two equal angles are half of the interior angle, then the included angle between the two extension lines is

$$\beta := 2\pi - \frac{2m\pi}{2m+1} - (\pi - \alpha) = \frac{3\pi}{2m+1}.$$

Furthermore, since the points u, v_2, v_{m+2} form an isosceles triangle, its base is the longest diagonal with length s, by the law of sines the distance between u and v_2 (or v_{m+2}) is

$$b := \frac{s \cdot \sin((\pi - \beta)/2)}{\sin\beta} = \frac{s}{2\sin(\beta/2)}.$$

On the other hand, if we extend the two sides K_1 and K_{m+2}, they will intersect at some point w, and the points u, v_2, w, v_{m+2} shape a kite. Thus, the included angle between the extension lines is

$$\gamma := 2\pi - \beta - 2(\pi - \alpha) = \frac{\pi}{2m+1}.$$

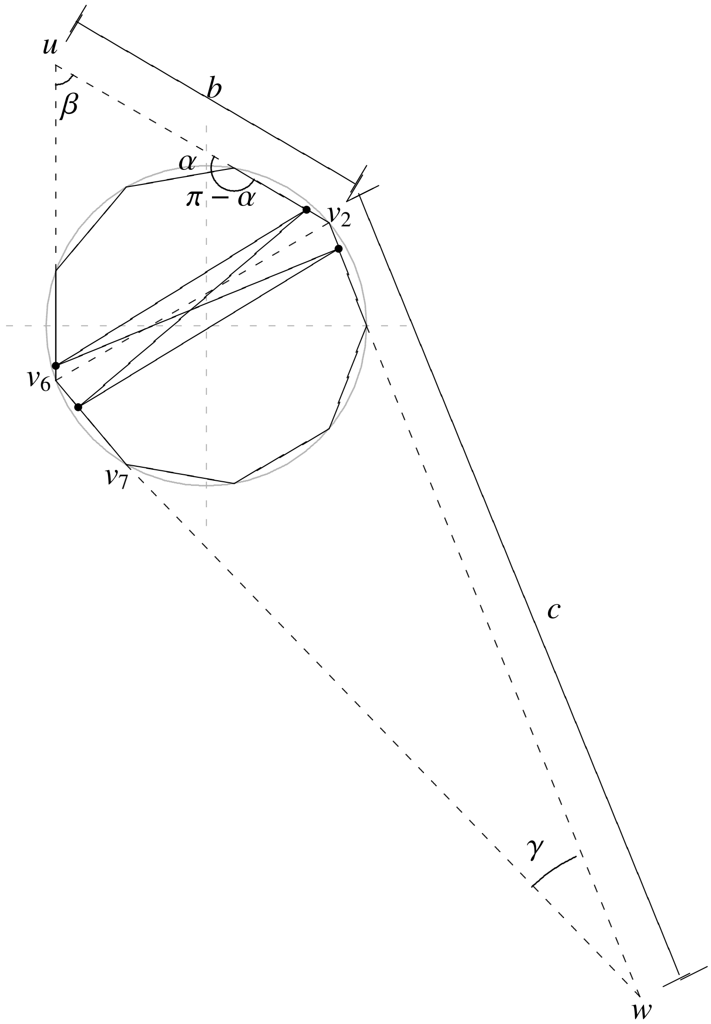

Figure 5.7: The geometric quantities ($m = 4$).

The distance between w and v_2 (or v_{m+2}) follows from the law of sines:

$$c := \frac{s \cdot \sin((\pi - \gamma)/2)}{\sin \gamma} = \frac{s}{2\sin(\gamma/2).}$$

Suppose that condition B_n holds. Analogous to the proof of Theorem 5.2, we denote for $i = 1, \ldots, 2m + 1$

$$Y_{2i-1} := \min_{X_j \; K_i^{(1)}} \; X_j - v_i \quad ,$$

$$Y_{2i} := \min_{X_j \; K_i^{(2)}} \; X_j - v_{i+1} \quad .$$

Thus, for each $k = 1, \ldots, 4m + 2$

$$\frac{n}{2(2m+1)} \cdot Y_k \xrightarrow{\mathcal{D}} E_k, \tag{5.12}$$

where $(E_k)_{k=1,\dots,4m+2}$ are independent and exponential distributed random variables with parameter $2/a$.

In the following, we represent the distances given in (5.7) and (5.8) with the help of $(Y_k)_{k=1,\dots,4m+2}$. For each $i = 1, \dots, m+1$ we have

$$
\begin{aligned}
& D_{rl}(i, m+i) \\
= \; & \Big((b - Y_{2i-1})^2 + (b - Y_{2(m+i-1)})^2 - 2(b - Y_{2i-1})(b - Y_{2(m+i-1)}) \cos\beta \Big)^{\frac{1}{2}} \\
= \; & \big(2b^2(1-\cos\beta) \big)^{\frac{1}{2}} - \frac{1}{2b} \big(2b^2(1-\cos\beta) \big)^{\frac{1}{2}} (Y_{2i-1} + Y_{2(m+i-1)}) \\
& \qquad + o_P(Y_{2i-1} + Y_{2(m+i-1)}) \\
= \; & s - \sin\frac{\beta}{2}(Y_{2i-1} + Y_{2(m+i-1)}) + o_P(Y_{2i-1} + Y_{2(m+i-1)}),
\end{aligned}
$$

where we take the first order Taylor approximation and use $b = s/(2\sin(\beta/2))$. Similarly, we have

$$
\begin{aligned}
& D_{rr}(i, m+i) \\
= \; & \Big(b^2 + (b - Y_{2i-1} - Y_{2(m+i)-1})^2 - 2b(b - Y_{2i-1} - Y_{2(m+i)-1}) \cos\beta \Big)^{\frac{1}{2}} \\
= \; & s - \sin\frac{\beta}{2}(Y_{2i-1} + Y_{2(m+i)-1}) + o_P(Y_{2i-1} + Y_{2(m+i)-1}),
\end{aligned}
$$

$$
\begin{aligned}
& D_{ll}(i, m+i) \\
= \; & \Big(b^2 + (b - Y_{2(i-1)} - Y_{2(m+i-1)})^2 - 2b(b - Y_{2(i-1)} - Y_{2(m+i-1)}) \cos\beta \Big)^{\frac{1}{2}} \\
= \; & s - \sin\frac{\beta}{2}(Y_{2(i-1)} + Y_{2(m+i-1)}) + o_P(Y_{2(i-1)} + Y_{2(m+i-1)}),
\end{aligned}
$$

$$
\begin{aligned}
& D_{lr}(i, m+i) \\
= \; & \Big((c - Y_{2(i-1)})^2 + (c - Y_{2(m+i)-1})^2 - 2(c - Y_{2(i-1)})(c - Y_{2(m+i)-1}) \cos\gamma \Big)^{\frac{1}{2}} \\
= \; & s - \sin\frac{\gamma}{2}(Y_{2(i-1)} + Y_{2(m+i)-1}) + o_P(Y_{2(i-1)} + Y_{2(m+i)-1}).
\end{aligned}
$$

Note that we use the notations $Y_0 := Y_{4m+2}$ and $E_0 := E_{4m+2}$. Analogously, we obtain for each $i = 1, \dots, m$

$$
\begin{aligned}
D_{rl}(i, m+i+1) &= s - \sin\frac{\gamma}{2}(Y_{2i-1} + Y_{2(m+i)}) + o_P(Y_{2i-1} + Y_{2(m+i)}), \\
D_{rr}(i, m+i+1) &= s - \sin\frac{\beta}{2}(Y_{2i-1} + Y_{2(m+i)+1}) + o_P(Y_{2i-1} + Y_{2(m+i)+1}), \\
D_{ll}(i, m+i+1) &= s - \sin\frac{\beta}{2}(Y_{2(i-1)} + Y_{2(m+i)}) + o_P(Y_{2(i-1)} + Y_{2(m+i)}), \\
D_{lr}(i, m+i+1) &= s - \sin\frac{\beta}{2}(Y_{2(i-1)} + Y_{2(m+i)+1}) + o_P(Y_{2(i-1)} + Y_{2(m+i)+1}).
\end{aligned}
$$

Applying (5.12) and the continuous mapping theorem yield the general expression of the limit law as follows:

Theorem 5.3. *Let $X_1, X_2, \dots$ be independent and uniformly distributed points on the sides of a $(2m+1)$-sided regular convex polygon, $m \geq 2$, with radius 1 and side length $a = 2\sin\frac{\pi}{2m+1}$. We then have*

$$
\begin{aligned}
&\frac{n}{2m+1}\cdot(s - D_n)\\
\xrightarrow{\mathcal{D}}\quad & 2\cdot\min_{\substack{i=1,\dots,m+1\\ j=1,\dots,m}}\Big\{\sin\frac{\beta}{2}\left(E_{2i-1}+E_{2(m+i-1)}\right),\ \sin\frac{\beta}{2}\left(E_{2i-1}+E_{2(m+i)-1}\right),\\
&\qquad \sin\frac{\beta}{2}\left(E_{2(i-1)}+E_{2(m+i-1)}\right),\ \sin\frac{\gamma}{2}\left(E_{2(i-1)}+E_{2(m+i)-1}\right),\\
&\qquad \sin\frac{\gamma}{2}\left(E_{2j-1}+E_{2(m+j)}\right),\ \sin\frac{\beta}{2}\left(E_{2j-1}+E_{2(m+j)+1}\right),\\
&\qquad \sin\frac{\beta}{2}\left(E_{2(j-1)}+E_{2(m+j)}\right),\ \sin\frac{\beta}{2}\left(E_{2(j-1)}+E_{2(m+j)+1}\right)\Big\}\\
=:\quad & 2\cdot E,
\end{aligned}
$$

where $\beta = 3\pi/(2m+1)$, $\gamma = \pi/(2m+1)$ and E_i, $i = 0, \dots, 4m+1$, are independent and exponential distributed random variables with parameter $2/a$.

Since E is the minimum of finitely many linear combinations of independent and exponential distributed random variables, its existence is ensured. However, the computation of the limit distribution function seems to be prohibitive.

Chapter 6

Largest distance in a support with major axes

In this chapter, we turn our attention to random points in a polytope or an ellipse. In the first section we investigate the limit law of the largest interpoint distance in case the underlying distribution is uniform in the unit square. After giving some simple polynomial bounds for the limit distribution function, we deduce the exact limit law by some asymptotic and geometric considerations. In subsequent sections we generalize the approach adopted in Section 6.1 in three directions. Firstly, we drop the restriction that the underlying distribution is uniform. Some general conditions on the density f of X_1 are given and several examples for f can be found in Section 6.2. Secondly, we tackle the general case of uniformly distributed points in a cube or a hypercube with dimension $d \geq 2$. In the final Section 6.4 the support of the uniform distribution of the random points is a regular convex polygon. Moreover, we treat the case of a uniform distribution in an ellipse. For the latter case we improve the lower bound for the limit law given in [4] by Appel, Najim and Russo.

6.1 Uniform distribution in the unit square

Suppose that $X_1, X_2, \ldots$ are independent with a uniform distribution in the unit square $[0,1]^2$. Let $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (1,1)$, $v_4 = (1,0)$ denote the four vertices. With $v_5 := v_1$, let K_i be the side connecting v_i and v_{i+1}, $i = 1, \ldots, 4$. Obviously, the largest interpoint distance $D_n := \max_{1 \leq i < j \leq n} \|X_i - X_j\|$ converges almost surely to the length $\sqrt{2}$ of the diagonals.

Choose a sequence of thresholds $\sqrt{2} - \varepsilon_n$ such that $\varepsilon_n \to 0$ as $n \to \infty$, and denote by

$$E_n(i,j) := \left\{ \|X_i - X_j\| > \sqrt{2} - \varepsilon_n \right\}, \qquad 1 \le i < j \le n,$$

the event that the distance between X_i and X_j exceeds the threshold. For each $x_0 \in [0,1]^2$ write

$$C_n(x_0) := \left\{ x \in [0,1]^2 : \|x - x_0\| > \sqrt{2} - \varepsilon_n \right\}$$

for the cap that contains the points having a distance to x_0 larger than $\sqrt{2} - \varepsilon_n$. Note that for large n, a necessary condition for an exceedance is that one of the endpoint lies in $C_n(v_k)$, $k = 1, 2$, and the other in $C_n(v_{k+2})$. By the central symmetry of the unit square, we study without loss of generality the area of $C_n(v_3)$.

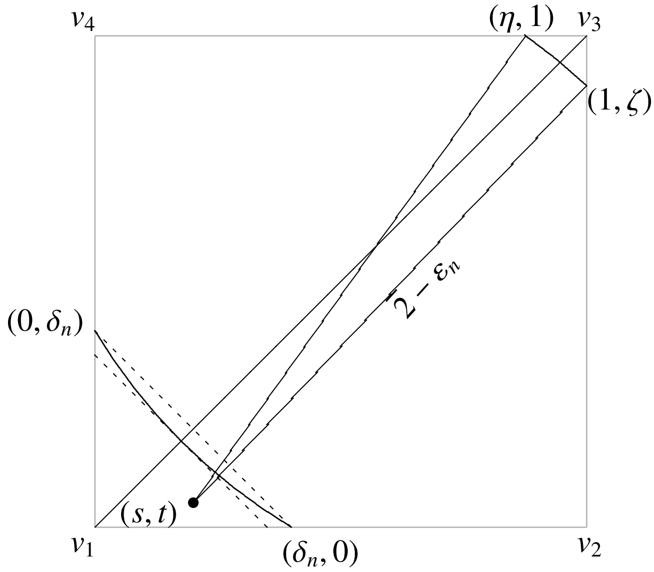

Figure 6.1: Illustration of the notations.

To obtain the cap $C_n(v_3)$ consider the circle with center v_3 and radius $\sqrt{2} - \varepsilon_n$ (see Figure 6.1). For sufficiently large n, this circle intersects the sides K_1 and K_4 at $(\delta_n, 0)$ and $(0, \delta_n)$ respectively, where $\delta_n = \sqrt{2}\varepsilon_n + \frac{1}{2}\varepsilon_n^2 + O\left(\varepsilon_n^3\right) \le \sqrt{2}\varepsilon_n + \varepsilon_n^2 =: \bar{\delta}_n$. Denote by $\bar{A}_n(v_3)$ the triangle formed by $(\bar{\delta}_n, 0)$, $(0, \bar{\delta}_n)$ and v_1, which contains $C_n(v_3)$ and has the area $\frac{1}{2}\bar{\delta}_n^2 = \frac{1}{2}(\sqrt{2}\varepsilon_n + \varepsilon_n^2)^2 = \varepsilon_n^2 + \sqrt{2}\varepsilon_n^3 + \frac{1}{2}\varepsilon_n^4$. Furthermore, the circle intersects the diagonal connecting v_1 and v_3 at $(\frac{\sqrt{2}}{2}\varepsilon_n, \frac{\sqrt{2}}{2}\varepsilon_n)$, the tangent to the circle at this point intersects K_1 and K_4 at $(\sqrt{2}\varepsilon_n, 0)$ and $(0, \sqrt{2}\varepsilon_n)$ respectively. The triangle $A_n(v_3)$ formed by these intersection points and v_1 is contained by $C_n(v_3)$ and has the area $\frac{1}{2}(\sqrt{2}\varepsilon_n)^2 = \varepsilon_n^2$. We therefore have $\text{area}(C_n(v_3)) \sim \varepsilon_n^2$ as $n \to \infty$. Moreover, the integral of a continuous function on $C_n(v_3)$ is asymptotically equal to the corresponding integral over $A_n(v_3)$.

The following lemma is also useful for later purposes:

Lemma 6.1. *There is a constant $C > 0$, such that for sufficiently large n*

$$\sup_{x=(s,t)\in C_n(v_3)} \left| \mathbb{P}\Big(\|X_1 - x\| > \sqrt{2} - \varepsilon_n \Big) - \left(\varepsilon_n - \frac{s+t}{\sqrt{2}} \right)^2 \right| \le C \cdot \varepsilon_n^3.$$

Proof. Since the density of X_1 is uniform over $[0,1]^2$, the probability in the formula above is equal to the area of the cap $C_n(x)$. Note that, for each $x = (s,t) \in C_n(x)$, we have $0 \le s,t \le \delta_n \le \sqrt{2}\varepsilon_n + \varepsilon_n^2$ for sufficiently large n. The circle with center x and radius $\sqrt{2} - \varepsilon_n$ intersects K_2 and K_3 at $(1,\zeta)$ and $(\eta, 1)$, respectively. (See Figure 6.1.) The area of $C_n(x)$ is bounded by the area of the triangle $A_n(\zeta)$ formed by v_3, $(1,\zeta)$, $(\zeta,1)$ and the area of the triangle $A_n(\eta)$ formed by v_3, $(\eta,1)$, $(1,\eta)$. In the following, we compute the area of the triangles area $(A_n(\zeta)) = \frac{1}{2}(1-\zeta)^2$ and area $(A_n(\eta)) = \frac{1}{2}(1-\eta)^2$. Some geometric considerations yield

$$\begin{aligned} 1-\zeta &= 1 - t - \sqrt{(\sqrt{2}-\varepsilon_n)^2 - (1-s)^2}, \\ 1-\eta &= 1 - s - \sqrt{(\sqrt{2}-\varepsilon_n)^2 - (1-t)^2}. \end{aligned}$$

We have on one hand

$$\sqrt{(\sqrt{2}-\varepsilon_n)^2 - (1-s)^2} \le \sqrt{(1-(\sqrt{2}\varepsilon_n - s))^2} = 1 - (\sqrt{2}\varepsilon_n - s)$$

as well as

$$\frac{1}{2}(1-\zeta)^2 \ge \frac{1}{2}\Big(\sqrt{2}\varepsilon_n - s - t\Big)^2 = \left(\varepsilon_n - \frac{s+t}{\sqrt{2}}\right)^2. \tag{6.1}$$

On the other hand, since $\varepsilon_n(s) := 2(\sqrt{2}\varepsilon_n - s) + \varepsilon_n^2 - s^2 \le 3\varepsilon_n$ for $0 \le s \le \sqrt{2}\varepsilon_n + \varepsilon_n^2$ and large enough n, we obtain by Taylor series expansions of ε_n around 0

$$\begin{aligned} \sqrt{(\sqrt{2}-\varepsilon_n)^2 - (1-s)^2} &= \sqrt{1 - 2(\sqrt{2}\varepsilon_n - s) + \varepsilon_n^2 - s^2} \\ &\ge 1 - \frac{1}{2}\left(2(\sqrt{2}\varepsilon_n - s) + \varepsilon_n^2 - s^2\right) - C_1\varepsilon_n^2 \\ &\ge 1 - (\sqrt{2}\varepsilon_n - s) - C_2\varepsilon_n^2, \end{aligned}$$

where C_1, C_2 are suitable positive constants that do not depend on s and t. It follows that

$$\frac{1}{2}(1-\zeta)^2 \le \frac{1}{2}\Big(\sqrt{2}\varepsilon_n - s - t + C_2\varepsilon_n^2\Big)^2 = \left(\varepsilon_n - \frac{s+t-C_2\varepsilon_n^2}{\sqrt{2}}\right)^2.$$

Together with (6.1), we have for n large enough

$$\left|\frac{1}{2}(1-\zeta)^2 - \left(\varepsilon_n - \frac{s+t}{\sqrt{2}}\right)^2\right| \le C_3\varepsilon_n^3,$$

where C_3 is some positive constant not depending on s and t. Analogously, we have

$$\left|\frac{1}{2}(1-\eta)^2 - \left(\varepsilon_n - \frac{s+t}{\sqrt{2}}\right)^2\right| \le C_4\varepsilon_n^3$$

for a suitable constant $C_4 > 0$ that does not depend on s and t. The lemma is proved. □

For $i, j, k, l \in \mathbb{N}$ we define the relations

$$\begin{aligned}(i,j) \prec (k,l) \quad &:\Longleftrightarrow \quad (i = k \text{ and } j < l) \text{ or } (i < k),\\ (i,j) \preceq (k,l) \quad &:\Longleftrightarrow \quad (i = k \text{ and } j \le l) \text{ or } (i < k),\end{aligned}$$

and set

$$\varepsilon_n := \varepsilon_n(t) := t \cdot n^{-1/2} \tag{6.2}$$

for $t > 0$. By the inclusion-exclusion principle we have

$$\begin{aligned} p_n(t) &:= \mathbb{P}\Big(n^{1/2}\cdot(\sqrt{2} - D_n) \le t\Big)\\ &= \mathbb{P}\Big(D_n > \sqrt{2} - \varepsilon_n\Big)\\ &= \mathbb{P}\Big(\bigcup_{1\le i<j\le n} E_n(i,j)\Big)\\ &= \sum_{\nu=1}^{n} (-1)^{\nu-1} S_{\nu,n}(t) \end{aligned}$$

where

$$S_{\nu,n}(t) := \sum_{\substack{(1,2)\preceq(i_1,j_1)\prec\ldots\\ \prec(i_\nu,j_\nu)\preceq(n-1,n)}} \mathbb{P}\left(E_n(i_1,j_1)\cap\ldots\cap E_n(i_\nu,j_\nu)\right).$$

In the following we give some bounds by using the Bonferroni inequalities.

Conditioning on $X_1 = x$ for a $x \in \bigcup_{i=1}^{4} C_n(v_i)$ and approximating the region of integration $C_n(v_3)$ by the triangle $A_n(v_3)$, Lemma 6.1 yields for $n \to \infty$

$$\begin{aligned}
\mathbb{P}(E_n(1,2)) &\sim 4 \cdot \int_{s=0}^{\sqrt{2}\varepsilon_n} \int_{t=0}^{\sqrt{2}\varepsilon_n - s} \left(\varepsilon_n - \frac{s+t}{\sqrt{2}}\right)^2 dt\, ds = \frac{2}{3}\varepsilon_n^4, \\
\mathbb{P}(E_n(1,2) \cap E_n(1,3)) &\sim 4 \cdot \int_{s=0}^{\sqrt{2}\varepsilon_n} \int_{t=0}^{\sqrt{2}\varepsilon_n - s} \left(\varepsilon_n - \frac{s+t}{\sqrt{2}}\right)^4 dt\, ds = \frac{4}{15}\varepsilon_n^6.
\end{aligned}$$

Plugging (6.2) into the expressions, we obtain

$$S_{1,n}(t) = \sum_{1 \le i<j \le n} \mathbb{P}(E_n(i,j)) = \binom{n}{2} \mathbb{P}(E_n(1,2)) \longrightarrow \frac{t^4}{3} =: S_1(t) \tag{6.3}$$

as well as

$$\begin{aligned}
S_{2,n}(t) &= \sum_{\substack{(1,2) \le (i,j) < \\ (k,l) \le (n-1,1)}} \mathbb{P}(E_n(i,j) \cap E_n(k,l)) \\
&= \binom{n}{3} \cdot 3 \cdot \mathbb{P}(E_n(1,2) \cap E_n(1,3)) + \binom{n}{4} \cdot 3 \cdot \mathbb{P}(E_n(1,2))^2 \\
&\longrightarrow \left(\frac{2t^6}{15} + \frac{t^8}{18}\right) =: S_2(t)
\end{aligned} \tag{6.4}$$

as $n \to \infty$. Thus, the Bonferroni inequalities yield

$$S_1(t) - S_2(t) \le \liminf_{n\to\infty} p_n(t) \le \limsup_{n\to\infty} p_n(t) \le S_1(t).$$

Similarly, we have for $n \to \infty$

$$S_{3,n}(t) \longrightarrow \left(\frac{83t^8}{1260} + \frac{2t^{10}}{45} + \frac{t^{12}}{162}\right) =: S_3(t), \tag{6.5}$$

$$S_{4,n}(t) \longrightarrow \left(\frac{t^8}{140} + \frac{4t^{10}}{105} + \frac{583t^{12}}{18900} + \frac{t^{14}}{135} + \frac{t^{16}}{11944}\right) =: S_4(t). \tag{6.6}$$

We then get the following more precise bounds:

Theorem 6.2. *Let $X_1, X_2, \ldots$ be independent and uniformly distributed points in the unit square $[0,1]^2$. For $t > 0$, we have*

$$\begin{aligned}
S_1(t) - S_2(t) + S_3(t) - S_4(t) &\le \liminf_{n\to\infty} \mathbb{P}\left(n^{1/2} \cdot \left(\sqrt{2} - D_n\right) \le t\right) \\
&\le \limsup_{n\to\infty} \mathbb{P}\left(n^{1/2} \cdot \left(\sqrt{2} - D_n\right) \le t\right) \\
&\le S_1(t) - S_2(t) + S_3(t),
\end{aligned}$$

where S_1, S_2, S_3 and S_4 are given in (6.3)-(6.6).

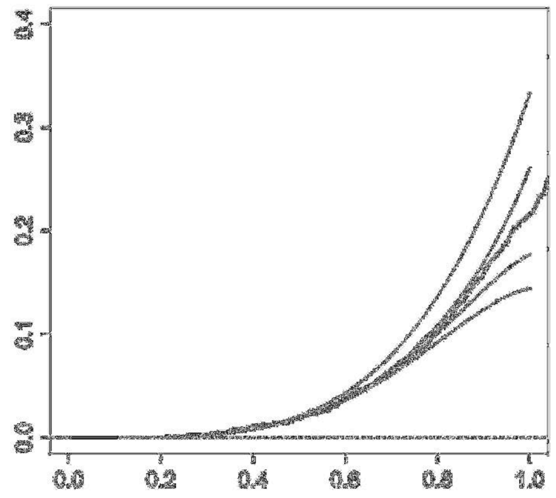

Figure 6.2: The thick curve is the EDF of $n^{1/2}(\sqrt{2} - D_n)$. The smooth curves are, from above to below, $S_1(t)$, $S_1(t) - S_2(t) + S_3(t)$, $S_1(t) - S_2(t) + S_3(t) - S_4(t)$ and $S_1(t) - S_2(t)$.

These polynomial lower and upper bounds for $p_n(t)$ become more and more precise by taking into account more $S_{v,n}$ s. However, the bounds are not always nondecreasing and do not take values in $[0, 1]$ for large t, so we have to restrict the choice of t to a bounded interval. Since the upper and lower bounds are both 0 for $t = 0$ and their difference increases in t, the approximation of $p_n(t)$ by these bounds is applicable only for small t (see Figure 6.2).

As noted above, only the points that are close to the vertices deserve attention for large n, and the candidates of the largest distance are the point connections which are close to one of the two diagonals. For reasons of symmetry, it suffices to consider the diagonal connecting $v_1 = (0, 0)$ and $v_3 = (1, 1)$.

In what follows, let Y_i denote the distance between the orthogonal projection of X_i onto the diagonal and the vertex v_1, $1 \le i \le n$. Let $U_n := \min_{1\le i\le n} Y_i$ and $V_n := \max_{1\le i\le n} Y_i$, thus $M_n := V_n - U_n$ is the sample range of Y_i, $i = 1, \dots, n$. With these notations we now give the limit law of M_n.

Lemma 6.3. *We have*

$$n^{1/2} \cdot (\sqrt{2} - M_n) \xrightarrow{\mathcal{D}} W_U + W_V$$

as $n \to \infty$, where W_U and W_V are independent Weibull distributed random variables with distribution function

$$\mathbb{P}(W_U \le t) = \mathbb{P}(W_V \le t) = 1 - e^{-\frac{t^2}{2}} =: G(t), \qquad t > 0.$$

Proof. Since the points are uniformly distributed in the unit square, $Y_1, Y_2, \ldots$ are i.i.d. with density

$$f_Y(t) = \begin{cases} 2t, & \text{if } 0 \le t < \frac{\sqrt{2}}{2}, \\ 2\sqrt{2} - 2t, & \text{if } \frac{\sqrt{2}}{2} \le t \le \sqrt{2}, \\ 0, & \text{otherwise,} \end{cases}$$

and distribution function

$$F_Y(t) = \begin{cases} 0, & \text{if } t < 0, \\ t^2, & \text{if } 0 \le t < \frac{\sqrt{2}}{2}, \\ 1 - (\sqrt{2} - t)^2, & \text{if } \frac{\sqrt{2}}{2} \le t \le \sqrt{2}, \\ 1, & \text{if } t > \sqrt{2}. \end{cases}$$

It follows that for each $t > 0$

$$\begin{aligned} \mathbb{P}\left(n^{1/2} \cdot U_n \le t\right) &= 1 - \mathbb{P}\left(U_n > tn^{-1/2}\right) \\ &= 1 - \left(1 - F_Y(tn^{-1/2})\right)^n \\ &= 1 - \left(1 - \frac{t^2}{n}\right)^n \xrightarrow{n \to \infty} 1 - e^{-t^2} = G(t) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}\left(n^{1/2} \cdot (\sqrt{2} - V_n) \le t\right) &= \mathbb{P}\left(V_n \ge \sqrt{2} - tn^{-1/2}\right) \\ &= \left(1 - F_Y\left(\sqrt{2} - tn^{-1/2}\right)\right)^n \\ &= 1 - \left(1 - \frac{t^2}{n}\right)^n \xrightarrow{n \to \infty} 1 - e^{-t^2} = G(t), \end{aligned}$$

which implies $n^{1/2} \cdot U_n \xrightarrow{\mathcal{D}} W_U$ and $n^{1/2} \cdot (\sqrt{2} - V_n) \xrightarrow{\mathcal{D}} W_V$ as $n \to \infty$ with W_U, W_V i.i.d. with the distribution function G. The result follows from the asymptotic independence of U_n, V_n and the continuous mapping theorem. □

By the convolution formula, the limit distribution function is

$$\begin{aligned} G * G(t) &= \int_0^t \int_0^s 2(s-u)e^{-(s-u)^2} 2ue^{-u^2} \, du \, ds \\ &= 1 - \left(e^{-t^2} + \sqrt{\frac{\pi}{2}}\, t\, e^{-t^2/2} \left(2\Phi(t) - 1\right)\right) \\ &=: 1 - H(t), \end{aligned} \tag{6.7}$$

where $\Phi(\cdot)$ denotes the distribution function of the standard normal law.

We shall now put an upper index in the notations above to denote the number of diagonals, where we number the diagonal connecting v_k and v_{k+2} by k, $k = 1, 2$. For instance, $Y_i^{(1)}$ $(i = 1, \ldots, n)$ are the distances between the projections of the points onto the diagonal connecting v_1 and v_3, $M_n^{(1)}$ denotes the sample range of $Y_i^{(1)}$, $1 \leq i \leq n$. The idea to obtain the limit law of D_n is to approximate D_n by the maximum of $M_n^{(1)}$ and $M_n^{(2)}$. Then, the main result of this section is stated as follows:

Theorem 6.4. *Let $X_1, X_2, \ldots$ be independent and uniformly distributed points in the unit square $[0, 1]^2$. We then have*

$$n^{1/2} \cdot \left(\sqrt{2} - D_n\right) \xrightarrow{\mathcal{D}} \min\left\{W_U^{(1)} + W_V^{(1)},\ W_U^{(2)} + W_V^{(2)}\right\}$$

as $n \to \infty$, where $W_U^{(k)}, W_V^{(k)}$, $k = 1, 2$, are independent Weibull distributed random variables with distribution function $G(t) = 1 - e^{-t^2}$, $t > 0$.

Note that the limit distribution function is given by

$$\begin{aligned} \lim_{n} \mathbb{P}\left(n^{1/2} \cdot \left(\sqrt{2} - D_n\right) \leq t\right) &= 1 - (1 - G * G(t))^2 \\ &= 1 - H(t)^2 \end{aligned}$$

for each $t > 0$ with $H(t)$ defined in (6.7).

Proof. Set $\varepsilon_n := \varepsilon_n(t) := tn^{-1/2}$ for $t > 0$. We now investigate the probability of an exceedance of D_n over the threshold $\sqrt{2} - \varepsilon_n$. On one hand, we have $D_n \geq \max\left\{M_n^{(1)}, M_n^{(2)}\right\}$, which implies

$$\mathbb{P}\left(D_n \geq \sqrt{2} - \varepsilon_n\right) \geq \mathbb{P}\left(\max\left\{M_n^{(1)}, M_n^{(2)}\right\} \geq \sqrt{2} - \varepsilon_n\right). \tag{6.8}$$

On the other hand, we assume that x_1 and x_2 are the endpoints of D_n which are closest to two opposite vertices, respectively. We move the diagonal connecting these vertices parallel to itself till it passes through one of the two endpoints x_1 and x_2 (see Figure 6.3). Without loss of generality let x_1 be the point on the parallel line. Denote by z_2 the orthogonal projection of x_2 onto the parallel line, then the distance between the orthogonal projections of x_1 and x_2 onto the diagonal is equal to $\|x_1 - z_2\|$.
Suppose that the inequality $\|x_1 - x_2\| \geq \sqrt{2} - \varepsilon_n$ holds. Note that a necessary condition for this is $\|x_2 - z_2\| \leq \sqrt{2}\delta_n \leq 2\varepsilon_n + \sqrt{2}\varepsilon_n^2$. Then, the Pythagorean

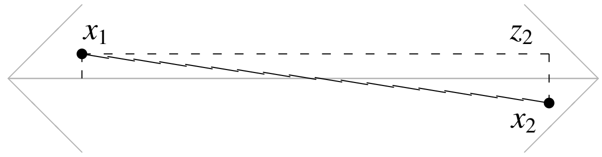

Figure 6.3: Geometric considerations in the proof of Theorem 6.4.

theorem yields for sufficiently large n

$$
\begin{aligned}
\|x_1 - z_2\| &= \sqrt{\|x_1 - x_2\|^2 - \|x_2 - z_2\|^2} \\
&\geq \sqrt{(\sqrt{2} - \varepsilon_n)^2 - (2\varepsilon_n + \sqrt{2}\varepsilon_n^2)^2} \\
&\geq \sqrt{2 - 2\sqrt{2}\varepsilon_n - 4\varepsilon_n^2} \\
&\geq \sqrt{2} - \varepsilon_n - 2\varepsilon_n^2,
\end{aligned}
$$

where we used a Taylor series expansion of ε_n around 0 in the last inequality. Since $\max\{M_n^{(1)}, M_n^{(2)}\} \geq \|x_1 - z_2\|$, we obtain

$$
\mathbb{P}\Big(D_n \geq \sqrt{2} - \varepsilon_n\Big) \leq \mathbb{P}\Big(\max\{M_n^{(1)}, M_n^{(2)}\} \geq \sqrt{2} - \varepsilon_n - 2\varepsilon_n^2\Big). \tag{6.9}
$$

It remains to show that the right-hand sides of (6.8) and (6.9) converge to the same limit as $n \to \infty$. By plugging $\varepsilon_n = tn^{-1/2}$ into the formula and using Lemma 6.3 we obtain on one hand

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\Big(\max\{M_n^{(1)}, M_n^{(2)}\} \geq \sqrt{2} - \varepsilon_n\Big) \\
&= \lim_{n\to\infty} \mathbb{P}\Big(n^{1/2}\big(\sqrt{2} - \max\{M_n^{(1)}, M_n^{(2)}\}\big) \leq t\Big) \\
&= \lim_{n\to\infty} \mathbb{P}\Big(\min\{n^{1/2}(\sqrt{2} - M_n^{(1)}),\ n^{1/2}(\sqrt{2} - M_n^{(2)})\} \leq t\Big) \\
&= \mathbb{P}\Big(\min\{W_U^{(1)} + W_V^{(1)},\ W_U^{(2)} + W_V^{(2)}\} \leq t\Big) \\
&= 1 - \mathbb{P}\Big(W_U^{(1)} + W_V^{(1)} > t\Big)^2 \\
&= 1 - H(t)^2,
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\Big(\max\{M_n^{(1)}, M_n^{(2)}\} \geq \sqrt{2} - \varepsilon_n - 2\varepsilon_n^2\Big) \\
&= \lim_{n\to\infty} \mathbb{P}\Big(n^{1/2}\big(\sqrt{2} - \max\{M_n^{(1)}, M_n^{(2)}\}\big) \leq t + 2t^2 n^{-1/2}\Big) \\
&= \mathbb{P}\Big(\min\{W_U^{(1)} + W_V^{(1)},\ W_U^{(2)} + W_V^{(2)}\} \leq t\Big) \\
&= 1 - H(t)^2.
\end{aligned}
$$

The theorem has been proved. □

Figure 6.4 shows a simulation of the EDF of $n^{1/2} \cdot (\sqrt{2} - D_n)$ with $n = 500$ points. The limit law is given as the dotted smooth curve. The simulation corroborates the result of Theorem 6.4.

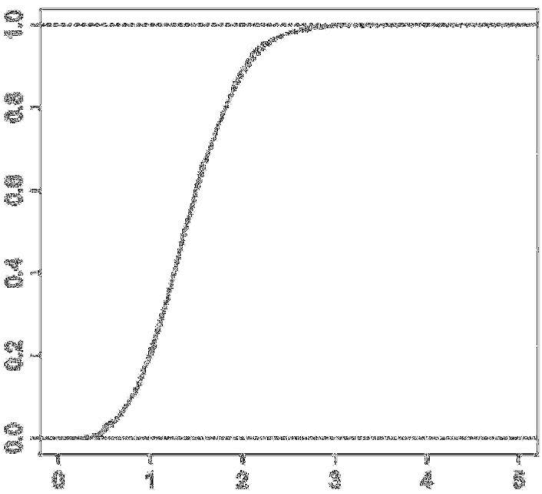

Figure 6.4: EDF of $n^{1/2} \cdot (\sqrt{2} - D_n)$, $n = 500$, and the limit distribution function $1 - H(t)^2$.

6.2 Non-uniform points in the unit square

The technique applied in the proof of Theorem 6.4 does not involve the uniform distribution of the random points. This assumption has been used in the proof of Lemma 6.3 to obtain the limit law of the sample range of the projections onto a diagonal. We shall now replace this assumption by a less restrictive one. Namely, if the sample range of the orthogonal projections of the points onto each diagonal has a limit law when suitably normalized, we can deduce the limit law of the largest interpoint distance.

To illustrate the situation, we first briefly discuss the univariate case. Let $X_1, X_2, \ldots$ be i.i.d. in the interval $[0, 1]$ with distribution function F. Since the asymptotic behavior of the sample range of the projections depends only on the local behavior of F at 0 and 1, we assume that F satisfies

$$\begin{aligned} F(s) &\sim as^{\alpha}, & s \to 0, \\ F(1-s) &\sim 1 - bs^{\beta}, & s \to 0, \end{aligned}$$

for some positive constant a, b and α, β. This implies that F behaves like a power function in the neighborhood of 0 and 1. Put $U_n := \min_{1\le i\le n} X_i$, $V_n := \max_{1\le i\le n} X_i$ and $M_n := V_n - U_n$. Then, $n^{1/\alpha}\cdot U_n$ and $n^{1/\beta}\cdot(1-V_n)$ converge as $n \to \infty$ to Weibull laws W_1 and W_2 with distribution functions $G_1(t) = 1-\exp\left(-at^\alpha\right)$ and $G_2(t) = 1-\exp\left(-bt^\beta\right)$, respectively. Moreover, the normalized extremes are asymptotically independent (see e.g. [17], Theorem 2.9.1). If $\alpha > \beta$, the asymptotic behavior of U_n dominates that of V_n, i.e., by using Sluzky's lemma we have for $n \to \infty$

$$n^{1/\alpha}\cdot(1-M_n) = n^{1/\alpha}\cdot(1-V_n) + n^{1/\alpha}\cdot U_n \xrightarrow{\mathcal{D}} W_1.$$

Similarly, for $\alpha < \beta$ we have

$$n^{1/\beta}\cdot(1-M_n) = n^{1/\beta}\cdot(1-V_n) + n^{1/\beta}\cdot U_n \xrightarrow{\mathcal{D}} W_2.$$

More interesting is the case $\alpha = \beta$, in which both U_n and V_n contribute to M_n in a non-negligible way. It follow from a result by Galambos [17] (Theorem 2.9.2) that for $n \to \infty$

$$n^{1/\alpha}\cdot(1-M_n) = n^{1/\alpha}\cdot(1-V_n) + n^{1/\alpha}\cdot U_n \xrightarrow{\mathcal{D}} W_1 + W_2.$$

Now, let $X_1, X_2, \dots$ be i.i.d. random points in the unit square with a density $f : [0,1]^2 \to \mathbb{R}_{\ge 0}$. Denote by K the support of $\mathbb{P}^{X_1}$. Assume that the Lebesgue measure of $[0,1]^2 \setminus K$ is zero and for each vertex there is an $l \in \mathbb{N}_0$ such that the l-th order (mixed) partial derivatives of f at the vertex are not all equal to zero. Denote by $Y_1^{(1)}, Y_2^{(1)}, \dots$ the distances from the orthogonal projections of the points onto the diagonal $\overline{v_1v_3}$ to v_1, and by $Y_1^{(2)}, Y_2^{(2)}, \dots$ the distances from the orthogonal projections of the points onto the diagonal $\overline{v_2v_4}$ to v_2. For each $k = 1, 2$, the random variables $Y_1^{(k)}, Y_2^{(k)}, \dots$ are i.i.d. in the interval $[0, \sqrt{2}]$ with the distribution function

$$F_Y^{(1)}(t) := \begin{cases} \displaystyle\int_{x=0}^{\sqrt{2}t}\int_{y=0}^{\sqrt{2}t-x} f(x,y)\,dy\,dx, & \text{if } 0 \le t \le \frac{\sqrt{2}}{2}, \\ \displaystyle 1 - \int_{x=1-\sqrt{2}t}^{1}\int_{y=2-\sqrt{2}t-x}^{1} f(x,y)\,dy\,dx, & \text{if } \frac{\sqrt{2}}{2} < t \le \sqrt{2}, \end{cases}$$

for $k = 1$ and

$$F_Y^{(2)}(t) := \begin{cases} \displaystyle\int_{x=1-\sqrt{2}t}^{1}\int_{y=0}^{x-1+\sqrt{2}t} f(x,y)\,dy\,dx, & \text{if } 0 \le t \le \frac{\sqrt{2}}{2}, \\ \displaystyle 1 - \int_{x=0}^{\sqrt{2}t}\int_{y=x+1-\sqrt{2}t}^{1} f(x,y)\,dy\,dx, & \text{if } \frac{\sqrt{2}}{2} < t \le \sqrt{2}, \end{cases}$$

for $k = 2$. Put $U_n^{(k)} := \min_{1\le i\le n} Y_i^{(k)}$ and $V_n^{(k)} := \max_{1\le i\le n} Y_i^{(k)}$, $k = 1, 2$. To study the asymptotic behavior of $U_n^{(k)}$ and $V_n^{(k)}$ as $n \to \infty$, we need more information about the behavior of f in the neighborhood of each vertex. For $k = 1, 2$, define

$$\alpha_k := \min\left\{ l \in \mathbb{N}_0 : \exists\, r \in \{0, \dots, l\} \text{ with } \frac{\partial^l}{\partial x^r \partial y^{l-r}} f(v_k) \neq 0 \right\}, \tag{6.10}$$

$$\beta_k := \min\left\{ l \in \mathbb{N}_0 : \exists\, r \in \{0, \dots, l\} \text{ with } \frac{\partial^l}{\partial x^r \partial y^{l-r}} f(v_{k+2}) \neq 0 \right\}, \tag{6.11}$$

where $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1, 1)$, $v_4 = (0, 1)$. The Taylor polynomial of order α_k at the point $v_k = (x_0, y_0)$, $k = 1, 2$, is

$$f(x, y) = \sum_{r=0}^{\alpha_k} \frac{(x - x_0)^r (y - y_0)^{\alpha_k - r}}{r!(\alpha_k - r)!} \cdot \frac{\partial^{\alpha_k}}{\partial x^r \partial y^{\alpha_k - r}} f(x_0, y_0),$$

so that by integrating we obtain

$$F_Y^{(k)}(t) \sim a_k t^{\alpha_k+2}, \qquad t \to 0, \tag{6.12}$$

for some constant $a_k > 0$. Similarly, the Taylor polynomial of order β_k at $v_{k+2} = (x_0, y_0)$, $k = 1, 2$, is

$$f(x, y) = \sum_{r=0}^{\beta_k} \frac{(x - x_0)^r (y - y_0)^{\beta_k - r}}{r!(\beta_k - r)!} \cdot \frac{\partial^{\beta_k}}{\partial x^r \partial y^{\beta_k - r}} f(x_0, y_0),$$

then

$$F_Y^{(k)}(\sqrt{2} - t) \sim 1 - b_k t^{\beta_k+2}, \qquad t \to 0, \tag{6.13}$$

with $b_k > 0$. For instance, if $f(0, 0) > 0$, then $F_Y^{(1)}(t) \sim f(0, 0)\, t^2$ as $t \to 0$. Using (6.12) and (6.13), we conclude that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(n^{1/(\alpha_k+2)} \cdot U_n^{(k)} \le t\right) &= 1 - \lim_{n\to\infty} \left(1 - F_Y^{(k)}\left(n^{-1/(\alpha_k+2)} t\right)\right)^n \\
&= 1 - \lim_{n\to\infty} \left(1 - a_k t^{\alpha_k+2} n^{-1}\right)^n \\
&= 1 - \exp\left(-a_k t^{\alpha_k+2}\right) =: G_U^{(k)}(t)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(n^{1/(\beta_k+2)} \cdot (\sqrt{2} - V_n^{(k)}) \le t\right) &= 1 - \lim_{n\to\infty} F_Y^{(k)}\left(\sqrt{2} - n^{-1/(\beta_k+2)} t\right)^n \\
&= 1 - \lim_{n\to\infty} \left(1 - b_k t^{\beta_k+2} n^{-1}\right)^n \\
&= 1 - \exp\left(-b_k t^{\beta_k+2}\right) =: G_V^{(k)}(t)
\end{aligned}$$

exist and are nondegenerate for $t > 0$, $k = 1, 2$. then the limit laws of the sample ranges $M_n^{(1)}$ and $M_n^{(2)}$ can be derived similarly to the univariate case. Namely, with different values for α_k and β_k, $M_n^{(k)}$ reduces to one of $U_n^{(k)}$ and $V_n^{(k)}$ in limit, $k = 1, 2$, and with $\alpha_k = \beta = k$, $M_n^{(k)}$ is the convolution of two asymptotically independent random variables with limit laws $G_U^{(k)}$ and $G_V^{(k)}$, respectively. The limit law of D_n is contained in the following theorem.

Theorem 6.5. *Let $X_1, X_2, \ldots$ be i.i.d. points in $[0,1]^2$ with density f and α_1, α_2, β_1, β_2 defined in (6.10) and (6.11), respectively. Denote $\gamma := \max\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. We then have as $n \to \infty$*

$$n^{1/(\gamma+2)} \cdot \left(\sqrt{2} - D_n\right) \xrightarrow{\mathcal{D}} \min\left\{W_U^{(1)} + W_V^{(1)},\ W_U^{(2)} + W_V^{(2)}\right\} =: Z,$$

where $W_U^{(k)}$ and $W_V^{(k)}$, $k = 1, 2$, are independent random variables and

(a) if $\alpha_k = \gamma$, then $W_U^{(k)}$ has the distribution function

$$\mathbb{P}\left(W_U^{(k)} \le t\right) = 1 - \exp\left(-a_k t^{\alpha_k+2}\right) = G_U^{(k)}(t), \qquad t > 0,$$

with some constant $a_k > 0$, else if $\alpha_k < \gamma$, then $W_U^{(k)} = 0$ almost surely;

(b) if $\beta_k = \gamma$, then $W_V^{(k)}$ has the distribution function

$$\mathbb{P}\left(W_V^{(k)} \le t\right) = 1 - \exp\left(-b_k t^{\beta_k+2}\right) = G_V^{(k)}(t), \qquad t > 0,$$

with some constant $b_k > 0$, else if $\beta_k < \gamma$, then $W_V^{(k)} = 0$ almost surely.

In the case $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma$, we obtain the distribution function of Z by a convolution formula, i.e. for $t > 0$

$$\mathbb{P}(Z \le t) = 1 - \left(1 - G_U^{(1)} * G_V^{(1)}(t)\right) \cdot \left(1 - G_U^{(2)} * G_V^{(2)}(t)\right).$$

Proof. Analogous to the proof of Theorem 6.4. □

To illustrate the theorem, we consider several examples.

Example 6.1. Let $X_1, X_2, \ldots$ be i.i.d. random points in $[0,1]^2$ with the pyramid-shaped density

$$f(x,y) := \begin{cases} 6y, & \text{if } y \in [0, \frac{1}{2}] \text{ and } x \in [y, 1-y), \\ 6(1-x), & \text{if } x \in (\frac{1}{2}, 1] \text{ and } y \in [1-x, x), \\ 6(1-y), & \text{if } y \in (\frac{1}{2}, 1] \text{ and } x \in (1-y, y], \\ 6x, & \text{if } x \in [0, \frac{1}{2}) \text{ and } y \in (x, 1-x] \end{cases}$$

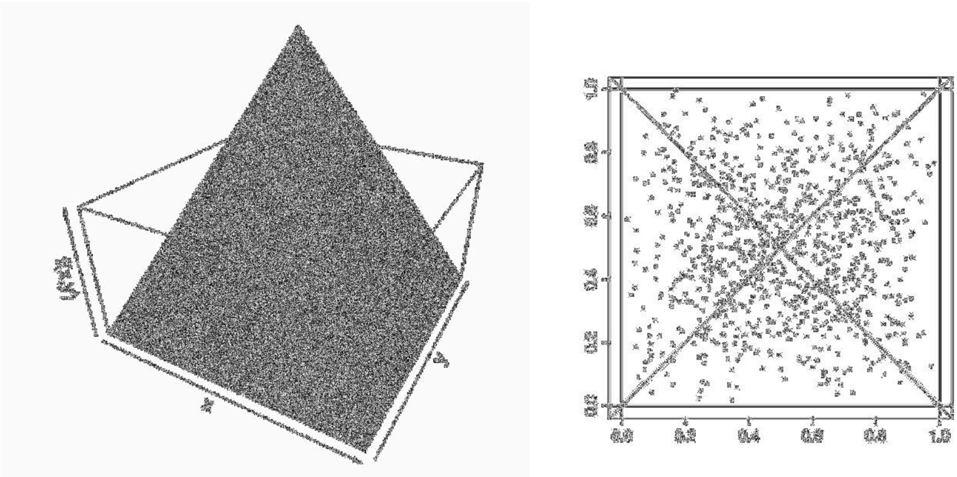

Figure 6.5: The density function of the pyramid-shaped point distribution and a sample with $n = 1000$ random points.

(see Figure 6.5). By symmetry, we only need to consider the sample range of the projections onto the diagonal $\overline{v_1 v_3}$. For each $t \in [0, \frac{\sqrt{2}}{2}]$, the probability $\mathbb{P}\left(Y_1^{(1)} \leq t\right)$ is equal to the volume of the tetrahedron with base area t^2 and height $f(t/\sqrt{2}, t/\sqrt{2})$. By straightforward algebra, we obtain for $k = 1, 2$

$$F_Y^{(k)}(t) \sim \sqrt{2}t^3, \qquad t \to 0.$$

It follows that for $t > 0$

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(n^{1/3} \cdot U_n^{(k)} \leq t\right) &= 1 - \lim_{n\to\infty} \left(1 - F_Y^{(k)}\left(n^{-1/3}t\right)\right)^n \\
&= 1 - \lim_{n\to\infty} \left(1 - \frac{\sqrt{2}t^3}{n}\right)^n \\
&= 1 - \exp\left(-\sqrt{2}t^3\right) =: G(t).
\end{aligned}$$

By symmetry of $F_Y^{(k)}$, $n^{1/3} \cdot (\sqrt{2} - V_n^{(k)})$ converges to the same limit. Hence, the limit law follows from Theorem 6.5:

$$n^{1/3} \cdot \left(\sqrt{2} - D_n\right) \xrightarrow{\mathcal{D}} \min\left\{W_U^{(1)} + W_V^{(1)},\ W_U^{(2)} + W_V^{(2)}\right\} =: Z$$

where $W_U^{(k)}$ and $W_V^{(k)}$, $k = 1, 2$, are i.i.d. Weibull random variables with the distribution function

$$G(t) = 1 - \exp\left(-\sqrt{2}t^3\right), \qquad t > 0.$$

Figure 6.6 shows the EDFs of $n^{1/3} \cdot (\sqrt{2} - D_n)$ with $n = 100$ (upper curve), $n = 1000$ (lower curve) points and the theoretical limit distribution function (thick smooth curve).

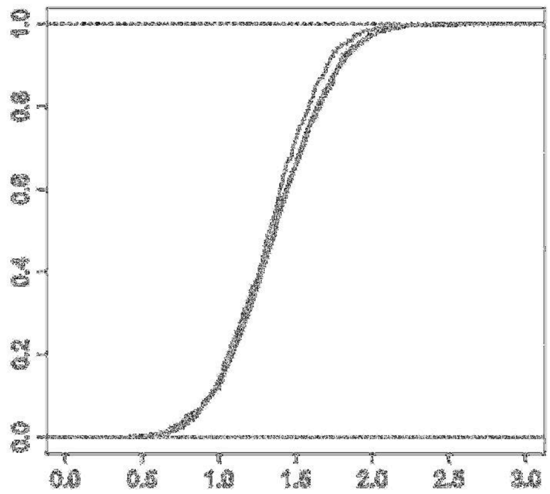

Figure 6.6: EDFs of $n^{1/3}\cdot(\sqrt{2}-D_n)$ with $n = 100$ (upper curve), $n = 1000$ (lower curve), and the limit distribution function for the pyramid-shaped density (thick smooth curve).

Example 6.2. Let $X_1, X_2, \ldots$ be i.i.d. random points in $[0,1]^2$ with the wedge-shaped density

$$f(x,y) := c-(2c-2)\cdot x, \qquad x,y\in[0,1],$$

for some positive constant $c\in[1,2]$ (see Figure 6.7). Note that the density is not symmetric with respect to the center of the unit square, but to the line $y = 1/2$. Moreover, $f(0,y) = c$ and $f(1,y) = 2-c$ are the maximum and the minimum of f, respectively. By some geometric considerations, the distribution function of $Y_i^{(1)}$, $i = 1,\ldots,n$, satisfies

$$\begin{aligned} F_Y^{(1)}(t) &= ct^2-\frac{\sqrt{2}}{3}(2c-2)t^3, && 0\le t<\frac{\sqrt{2}}{2},\\ F_Y^{(1)}(t) &= 1-(2-c)(\sqrt{2}-t)^2-\frac{\sqrt{2}}{3}(2c-2)(\sqrt{2}-t)^3, && \frac{\sqrt{2}}{2}\le t\le\sqrt{2}. \end{aligned}$$

Choosing $n^{1/2}$ as the normalizing factor for $U_n^{(1)}$, it follows that

$$\begin{aligned} \lim_{n\to\infty}\mathbb{P}\left(n^{1/2}\cdot U_n^{(1)}\le t\right) &= 1-\lim_{n\to\infty}\left(1-F_Y^{(1)}\left(n^{-1/2}t\right)\right)^n\\ &= 1-\lim_{n\to\infty}\left(1-\frac{ct^2}{n}+O\left(n^{-3/2}\right)\right)^n\\ &= 1-\exp\left(-ct^2\right) =: G_1(t) \end{aligned}$$

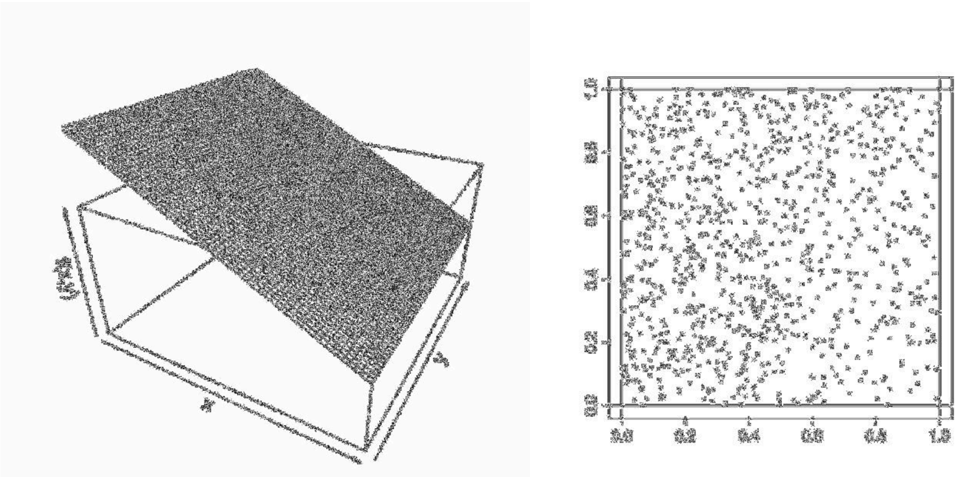

Figure 6.7: The density function of the wedge-shaped point distribution with $c = 3/2$ and a sample with $n = 1000$ random points.

for $t > 0$. To derive the limit law of $V_n^{(1)}$, we treat the cases $c = 2$ and $c < 2$ separately. If $c = 2$, i.e. if $f(1, y) = 0$, the squared term in $F_Y^{(1)}$ vanishes for $\frac{\sqrt{2}}{2} \leq t \leq \sqrt{2}$. We choose $n^{1/3}$ as the normalizing factor and obtain for $t > 0$

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\Big(n^{1/3} \cdot \big(\sqrt{2} - V_n^{(1)}\big) \leq t\Big) &= 1 - \lim_{n\to\infty} \Big(F_Y^{(1)}\big(\sqrt{2} - n^{-1/3}t\big)\Big)^n \\
&= 1 - \lim_{n\to\infty} \left(1 - \frac{2\sqrt{2}}{3} \cdot \frac{t^3}{n}\right)^n \\
&= 1 - \exp\left(-\frac{2\sqrt{2}}{3}t^3\right).
\end{aligned}$$

Since the asymptotic behavior of $V_n^{(1)}$ dominates that of $U_n^{(1)}$, the symmetry of f with respect to $y = 1/2$ and Theorem 6.5 yield

$$\lim_{n\to\infty} \mathbb{P}\Big(n^{1/3} \cdot \big(\sqrt{2} - D_n\big) \leq t\Big) = 1 - \exp\left(-\frac{2\sqrt{2}}{3}t^3\right).$$

If $c < 2$, choosing $n^{1/2}$ as the normalizing factor for $V_n^{(1)}$ yields for $t > 0$

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\Big(n^{1/2} \cdot \big(\sqrt{2} - V_n^{(1)}\big) \leq t\Big) &= 1 - \lim_{n\to\infty} \Big(F_Y^{(1)}\big(\sqrt{2} - n^{-1/2}t\big)\Big)^n \\
&= 1 - \lim_{n\to\infty} \left(1 - \frac{(2-c)t^2}{n} + O\big(n^{-3/2}\big)\right)^n \\
&= 1 - \exp\Big(-(2-c)t^2\Big) =: G_2(t).
\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \mathbb{P}\Big(n^{1/2} \cdot \big(\sqrt{2} - M_n^{(1)}\big) \leq t\Big) = G_1 * G_2(t).$$

By symmetry of f with respect to $y = 1/2$, $M_n^{(2)}$ has the same limit law as $M_n^{(1)}$. Hence, we have

$$n^{1/2} \cdot \left(\sqrt{2} - D_n \right) \xrightarrow{\mathcal{D}} \min\left\{ W_U^{(1)} + W_V^{(1)},\ W_U^{(2)} + W_V^{(2)} \right\} =: Z$$

where the distribution function of Z is

$$\mathbb{P}(Z \le t) = 1 - (1 - G_1 * G_2(t))^2, \qquad t > 0.$$

A simulation of the EDF s of $n^{1/2} \cdot (\sqrt{2} - D_n)$ with $n = 100$ (upper curve) and $n = 1000$ (lower curve) points can be found in Figure 6.8. Note that $c = 1$ yields the uniform distribution on $[0, 1]^2$. In this case we have $G_1(t) = G_2(t) = 1 - \exp(-t^2)$ and thus the limit law of Theorem 6.4.

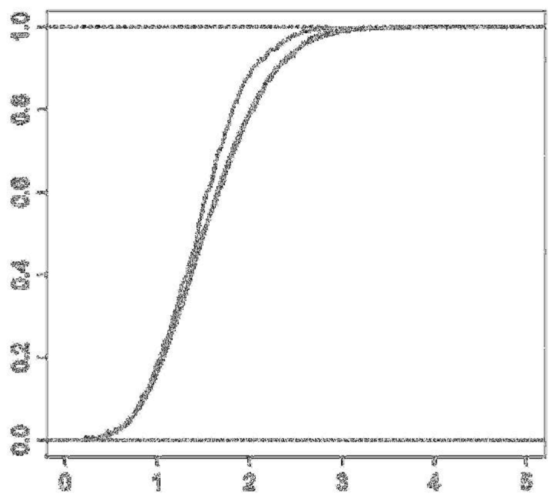

Figure 6.8: EDF s of $n^{1/2} \cdot (\sqrt{2} - D_n)$ with $n = 100$ (upper curve), $n = 1000$ (lower curve) and the limit distribution function for the wedge-shaped density with $c = 3/2$ (thick smooth curve).

Example 6.3. Let $X_1, X_2, \ldots$ be i.i.d. random points in $[0, 1]^2$ with the paraboloidal-cup-shaped density

$$f(x, y) := c\left(x - \frac{1}{2}\right)^2 + c\left(y - \frac{1}{2}\right)^2 + 1 - \frac{c}{6}, \qquad x, y \in [0, 1], \tag{6.14}$$

for some constant $c \in [0, 6]$. Note that the case $c = 0$ yields the uniform distribution on $[0, 1]^2$. If $c = 6$ the curvature is maximal and f vanishes at the center of the unit square. A plot of f with $c = 6$ and an example of a point sample can be found in Figure 6.9. The density is symmetric with respect to $(\frac{1}{2}, \frac{1}{2})$ and attains its maximal value $1 + c/3$ at the four vertices. For $t \in [0, \frac{\sqrt{2}}{2}]$

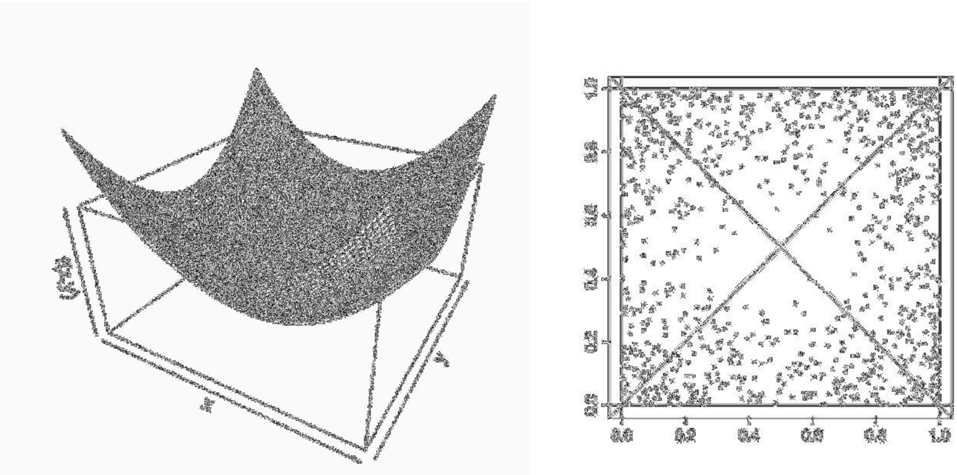

Figure 6.9: The density (6.14) with $c = 6$ and a sample with $n = 1000$ random points.

we obtain the distribution function $F_Y^{(k)}$, $k = 1, 2$, by integrating f on the triangle formed by $(\sqrt{2}t, 0)$, $(0, \sqrt{2}t)$ and v_1. The result is

$$F_Y^{(k)}(t) = \left(1 + \frac{c}{3}\right)t^2 - \frac{2\sqrt{2}}{3}ct^3 + \frac{2}{3}ct^4 \sim \left(1 + \frac{c}{3}\right)t^2, \qquad t \downarrow 0.$$

By symmetry of f, $F_Y^{(k)}$ is symmetric with respect to $\sqrt{2}/2$. For $t \in (\frac{\sqrt{2}}{2}, \sqrt{2}]$ we thus have for $k = 1, 2$

$$F_Y^{(k)}(t) \;=\; 1 - F_Y^{(k)}\left(\sqrt{2} - t\right).$$

Choosing $n^{1/2}$ as the normalizing factor for both of $U_n^{(k)}$ and $V_n^{(k)}$, $k = 1, 2$, we then have for $t > 0$

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(n^{1/2} \cdot U_n^{(k)} \le t\right) &= 1 - \lim_{n\to\infty} \left(1 - F_Y^{(k)}\left(n^{-1/2}t\right)\right)^n \\
&= 1 - \lim_{n\to\infty} \left(1 - \left(1 + \frac{c}{3}\right)\frac{t^2}{n}\right)^n \\
&= 1 - \exp\left(-\left(1 + \frac{c}{3}\right)t^2\right) \;=:\; G(t)
\end{aligned}$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(n^{1/2} \cdot \left(\sqrt{2} - V_n^{(k)}\right) \le t\right) = 1 - \lim_{n\to\infty} \left(F_Y^{(k)}\left(\sqrt{2} - n^{-1/2}t\right)\right)^n = G(t).$$

Consequently, we derive the limit law by Theorem 6.5:

$$n^{1/2} \cdot \left(\sqrt{2} - D_n\right) \xrightarrow{\mathcal{D}} \min\left\{W_U^{(1)} + W_V^{(1)},\ W_U^{(2)} + W_V^{(2)}\right\} =: Z$$

where $W_U^{(k)}$ and $W_V^{(k)}$, $k = 1, 2$, are i.i.d. Weibull random variables with the distribution function

$$G(t) = 1 - \exp\left(-(1 + c/3)\, t^2\right), \qquad t > 0.$$

Figure 6.10 shows a simulation of the EDF s of $n^{1/2} \cdot (\sqrt{2} - D_n)$ with $n = 100$ (lower curve) and $n = 1000$ (upper curve) points.

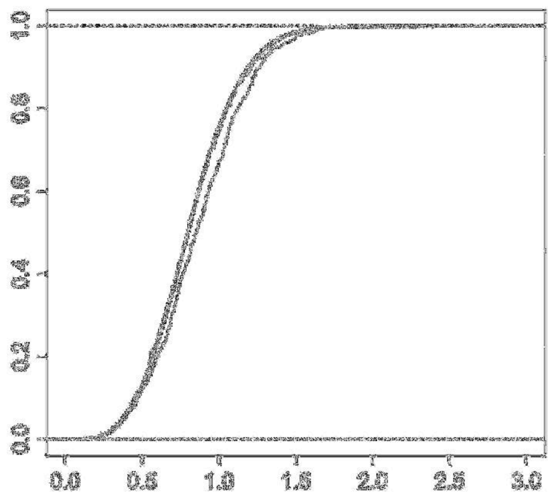

Figure 6.10: EDF s of $n^{1/2} \cdot (\sqrt{2} - D_n)$ with $n = 100$ (lower curve) and $n = 1000$ (upper curve) as well as the limit distribution function for the density (6.14) with $c = 6$ (thick smooth curve).

Example 6.4. As a last example we discuss the case with a paraboloidal-cap-shaped density. Let $X_1, X_2, \ldots$ be i.i.d. random points with density

$$f(x, y) := -c\left(x - \frac{1}{2}\right)^2 - c\left(y - \frac{1}{2}\right)^2 + 1 + \frac{c}{6}, \qquad x, y \in [0, 1], \qquad (6.15)$$

for some constant $c \in [0, 3]$. If $c = 0$, the density is a uniform distribution in $[0, 1]^2$ and, if $c = 3$, the curvature is maximal and the density vanishes at the four vertices (see Figure 6.11). The density is symmetric with respect to the center of the unit square and attains its minimum $1 - c/3$ at each of the vertices.

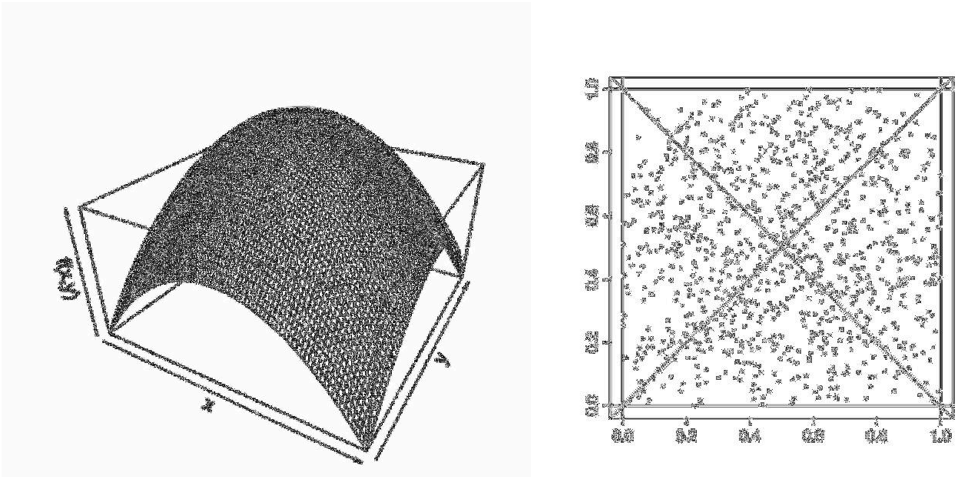

Figure 6.11: The density function (6.15) with $c = 3$ and a sample with $n = 1000$ random points.

We obtain the distribution function $F_Y^{(k)}$ for $t \in [0, \frac{\sqrt{2}}{2}]$ by integrating f over the triangle formed by $(\sqrt{2}t, 0)$, $(0, \sqrt{2}t)$ and v_1. Straightforward calculations yield for $k = 1, 2$

$$F_Y^{(k)}(t) = \left(1 - \frac{c}{3}\right)t^2 + \frac{2\sqrt{2}}{3}ct^3 - \frac{2}{3}ct^4, \tag{6.16}$$

and by symmetry,

$$F_Y^{(k)}(t) = 1 - F_Y^{(k)}\left(\sqrt{2} - t\right), \qquad t \in \left[\frac{\sqrt{2}}{2}, \sqrt{2}\right].$$

Notice that the squared term in (6.16) vanishes for the case $c = 3$, which requires a different choice of the normalizing factor for $U_n^{(k)}$ and $V_n^{(k)}$, $k = 1, 2$.

For $c \in [0, 3)$ we choose $n^{1/2}$ as the normalizing factor for both of the extremes. Then, for each $t > 0$

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(n^{1/2} \cdot U_n^{(k)} \le t\right) &= \lim_{n\to\infty} \mathbb{P}\left(n^{1/2} \cdot \left(\sqrt{2} - V_n^{(k)}\right) \le t\right) \\
&= 1 - \lim_{n\to\infty} \left[1 - F_y^{(1)}\left(n^{-1/2}t\right)\right]^n \\
&= 1 - \lim_{n\to\infty} \left[1 - \left(1 - \frac{c}{3}\right)\frac{t^2}{n} + O\left(n^{-3/2}\right)\right]^n \\
&= 1 - \exp\left(-\left(1 - \frac{c}{3}\right)t^2\right) =: G(t).
\end{aligned}$$

Applying Theorem 6.5 we get the limit law of D_n as follows:

$$n^{1/2} \cdot \left(\sqrt{2} - D_n\right) \xrightarrow{\mathcal{D}} \min\left\{W_U^{(1)} + W_V^{(1)},\ W_U^{(2)} + W_V^{(2)}\right\},$$

where $W_U^{(k)}$ and $W_V^{(k)}$, $k = 1, 2$, are i.i.d. Weibull random variables with the distribution function

$$G(t) = 1 - \exp\left(-\,(1 - c/3)\,t^2\right), \qquad t > 0.$$

Figure 6.12 shows that the approximation of limit law by the EDF becomes

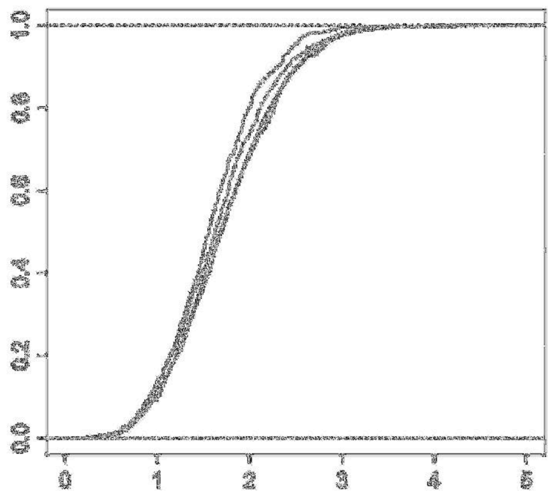

Figure 6.12: EDF s of $n^{1/2} \cdot (\sqrt{2} - D_n)$ for $n = 200$ (upper curve), $n = 1000$ (middle curve) and $n = 5000$ (lower curve) together with the limit distribution function (thick smooth curve) for the density (6.15), $c = 1$.

better with increasing n.

Now, for $c = 3$ we choose $n^{1/3}$ as the normalizing factor. For $t > 0$ it follows that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(n^{1/3} \cdot U_n^{(k)} \le t\right) &= \lim_{n\to\infty} \mathbb{P}\left(n^{1/3} \cdot \left(\; - V_n^{(k)}\right) \le t\right) \\
&= 1 - \lim_{n\to\infty} \left[1 - F_y^{(k)}\left(n^{-1/3} t\right)\right]^n \\
&= 1 - \lim_{n\to\infty} \left[1 - \left(2\sqrt{2}c\right)\frac{t^3}{n} + O\left(n^{-2}\right)\right]^n \\
&= 1 - \exp\left(-\left(2\sqrt{2}c\right)t^3\right) =: G(t).
\end{aligned}$$

Theorem 6.5 yields the limit law

$$n^{1/3} \cdot \left(\sqrt{2} - D_n\right) \xrightarrow{\mathcal{D}} \min\left(W_U^{(1)} + W_V^{(1)},\; W_U^{(2)} + W_V^{(2)}\right)$$

where $W_U^{(k)}$ and $W_V^{(k)}$, $k = 1, 2$, are i.i.d. Weibull random variables with the distribution function

$$G(t) = 1 - \exp\left(-2\sqrt{2}ct^3\right), \qquad t > 0.$$

Figure 6.13 illustrates an approximation of the limit law by the EDF of $n^{-1/3}$.

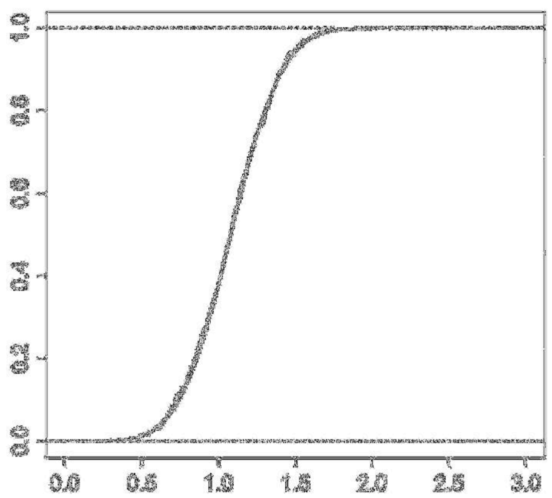

Figure 6.13: EDF of $n^{1/3} \cdot (\sqrt{2} - D_n)$, $n = 1000$, and the limit distribution function (thick smooth curve) for the density (6.15) with $c = 3$.

$(\sqrt{2} - D_n)$, $n = 1000$, for the paraboloidal-cap-shaped distribution with $c = 3$.

We now discuss the influence of the parameter c on the speed of convergence. In Figure 6.14 there is a comparison of simulations with $c = 0, 1, 2$ and 3 and $n = 1000$ points, respectively. As mentioned, for $c = 0$ the distribution is uniform, and the limit law agrees with that given in Theorem 6.4, with the convergence rate $n^{1/2}$. For $c \in (0, 3)$ the density is strictly positive on the unit square and the convergence rate is $n^{1/2}$. We note that the speed of convergence seems to decrease with growing c. For $c = 3$ the density vanishes at the four vertices of the unit square, and the rate of convergence is $n^{1/3}$. We see that the EDF with $n = 1000$ points resembles the limit law very closely.

6.3 Generalizations: The unit d-cube

In this section, let $X_1, X_2, \ldots$ be uniformly distributed points in the unit hypercube $[0, 1]^d$, $d \geq 2$. Such unit hypercube has 2^d vertices with coordinates equal

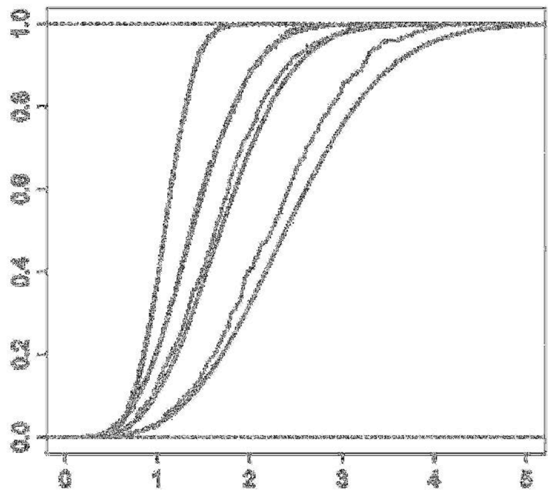

Figure 6.14: EDFs ($n = 1000$) and limit distribution functions (thick smooth curves) of D_n for the density (6.15) with $c = 3$, $c = 0$, $c = 1$ and $c = 2$ (from above to below).

to 0 or 1 and $2^{d-1}d$ edges with length 1. There are 2^{d-1} space diagonals with the maximal length of $\sqrt{d}$, to which D_n converges almost surely. The key part to derive the limit law of D_n is to determine the limit law of the sample range of the orthogonal projections of $X_1, \dots, X_n$ onto a space diagonal.

Generalizing Theorem 6.4, we shall prove the following result.

Theorem 6.6. *Let $X_1, X_2, \dots$ be independent and uniformly distributed in the unit hypercube $[0,1]^d$, $d \geq 2$. We then have*

$$n^{1/d} \cdot \left(\sqrt{d} - D_n\right) \xrightarrow{\mathcal{D}} \min_{1 \leq k \leq 2^{d-1}} \left(W_U^{(k)} + W_V^{(k)} \right) =: Z$$

as $n \to \infty$, where $W_U^{(k)}$, $W_V^{(k)}$, $k = 1, \dots, 2^{d-1}$, are independent Weibull distributed random variables with distribution function

$$G(t) = 1 - \exp\left(-\frac{d^{d/2}}{d!} t^d\right), \qquad t > 0.$$

Moreover, the limit distribution function is

$$\mathbb{P}(Z \leq t) = 1 - \{1 - G * G(t)\}^{2^{d-1}}.$$

Proof. We consider the space diagonal connecting $(0, \dots, 0)^T$ and $(1, \dots, 1)^T$. Let $Y_1, Y_2, \dots$ denote the distances between the orthogonal projections of the points onto this diagonal and the vertex $(0, \ldots, 0)^T$. Note that $Y_1, Y_2, \ldots$ are i.i.d. and take values in the interval $[0, \sqrt{d}]$. We derive the distribution function of Y_1 by some geometric considerations. Choose a $t > 0$ sufficiently small and consider a $(d-1)$-dimensional hyperplane orthogonal to the diagonal and having a distance t from the vertex $(0, \ldots, 0)^T$. (The chosen t is so small that the orthogonal hyperplane intersects only the edges of the hypercube with $(0, \ldots, 0)^T$ as one of the endpoints.) This hyperplane cuts off a corner from the hypercube. The cut-off part is a d-dimensional simplex formed by the vertex $v_1 := (0, \ldots, 0)^T$ and the d intersection points $w_1, \ldots, w_d$ of the hyperplane and the edges. We first determine the common distance s between v_1 and w_i, $i = 1, \ldots, d$. This simplex can be regarded as a pyramid with apex v_1 and base formed by $w_1, \ldots, w_d$. Since all adjacent edges at v_1 are pairwise orthogonal, the base is a $(d-1)$-dimensional regular simplex with edge length $\sqrt{2}s$. Moreover, the circumradius of the base is $r = \sqrt{s^2 - t^2}$. It is well-known that $d^2 r^2$ is greater than or equal to the sum of squared lengths of the edges (see e.g. [9]) and, equality holds if and only if the simplex is regular. We then have

$$d^2(s^2 - t^2) = \binom{d}{2} \left(\sqrt{2}s\right)^2 = d(d-1)s^2$$

and hence $s = \sqrt{d}t$. Thus, w_j is the vector with the j-th coordinate equal to $\sqrt{d}t$ and other coordinates equal to 0, $j = 1, \ldots, d$. Since the points are uniformly distributed, the volume of the simplex is the probability $\mathbb{P}(Y_1 \leq t)$. The well-known formula yields

$$\begin{aligned} \mathbb{P}(Y_1 \leq t) &= \frac{1}{d!} \det(w_1 - v_1, \ldots, w_d - v_1) \\ &= \frac{1}{d!} \cdot \left(\sqrt{d}t\right)^d = \frac{d^{d/2}}{d!} t^d. \end{aligned}$$

Putting $U_n := \min_{1 \leq i \leq n} Y_i$ and $V_n := \max_{1 \leq i \leq n} Y_i$, we have for each $t > 0$

$$\begin{aligned} \lim_{n} \mathbb{P}\left(n^{1/d} \cdot U_n \leq t\right) &= \lim_{n} \left[1 - \mathbb{P}\left(U_n > n^{-1/d} t\right)\right] \\ &= \lim_{n} \left[1 - \left(1 - \frac{d^{d/2}}{d!} \cdot \frac{t^d}{n}\right)^n\right] \\ &= 1 - \exp\left(-\frac{d^{d/2}}{d!} t^d\right) = G(t), \end{aligned}$$

and, by symmetry, we obtain similarly

$$\lim_{n} \mathbb{P}\left(n^{1/d} \cdot \left(\sqrt{d} - V_n\right) \leq t\right) = G(t), \qquad t > 0.$$

Hence, the sample range M_n of $Y_1, \ldots, Y_n$ has the following asymptotic distribution:

$$\lim_{n\to\infty} \mathbb{P}\left(n^{1/d} \cdot \left(\sqrt{d} - M_n\right) \le t\right) = G * G(t), \qquad t > 0.$$

The rest of the proof is analogous to the proof of Theorem 6.4. □

The next result is given to exemplify the theorem above.

Corollary 6.7. *Let $X_1, X_2, \ldots$ be independent and uniformly distributed points in the unit cube $[0,1]^3$. We then have*

$$n^{1/3} \cdot \left(\sqrt{3} - D_n\right) \xrightarrow{\mathcal{D}} \min_{k=1,2,3,4} \left(W_U^{(k)} + W_V^{(k)}\right) =: Z$$

as $n \to \infty$, where $W_U^{(k)}$, $W_V^{(k)}$, $k = 1,2,3,4$, are independent Weibull distributed with distribution function $G(t) = 1 - \exp\left(-\frac{\sqrt{3}}{2}t^3\right)$, $t > 0$. More precisely, the limit distribution function is

$$\begin{aligned}\mathbb{P}(Z \le t) \;=\; 1 - \Bigg(& \frac{1}{2}\exp\left(-\frac{\sqrt{3}}{2}t^3\right) + \frac{3^{3/4}\sqrt{2\pi}}{72}t^{-3/2}\left(9t^3 + 4\sqrt{3}\right) \\ & \cdot \exp\left(-\frac{\sqrt{3}}{8}t^3\right)\left(2\Phi\left(\frac{3^{3/4}}{2}t^{3/2}\right) - 1\right)\Bigg)^4,\end{aligned}$$

$t > 0$, where $\Phi(\cdot)$ is the distribution function of the standard normal distribution.

Figure 6.15 shows an approximation of the limit law in Corollary 6.7 by the EDF of $n^{1/3} \cdot (\sqrt{3} - D_n)$, $n = 500$.

6.4 Generalizations: Polygons and ellipse

In this section, we shall turn back to the 2-dimensional situation. We note that there are three crucial points in our method. The first one is that the support of the point distribution has finitely many major axes with disjoint sets of endpoints, that ensures the independence of the sample ranges of the orthogonal projections onto each axis. Secondly, the point distribution has a smooth density, so that the maximum and the minimum of the distances between the orthogonal projections onto an axis and one of the endpoints converge weakly to nondegenerate distributions, respectively. The last crucial point is that D_n is

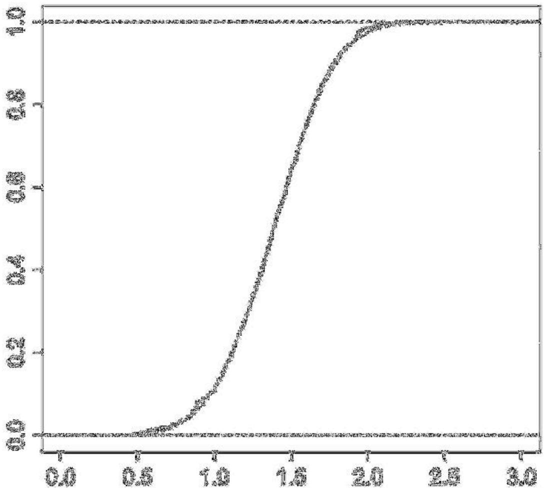

Figure 6.15: EDF of $n^{1/3} \cdot (\sqrt{3} - D_n)$, $n = 500$, and the limit distribution function (dotted smooth curve), $d = 3$.

asymptotically equal to the maximum of all sample ranges of the projections onto the axes. In the following, we shall get rid of the restriction of the unit square.

We consider a regular convex polygon with $2m$ sides, $m \in \mathbb{N}$, which has unit area, interior angle $\beta = \pi - \pi/m$ and m major diagonals with length $s = 2/\sqrt{m \sin(\pi/m)}$. For a uniform point sample in such support, we have the following result:

Theorem 6.8. *Let $X_1, X_2 \ldots$ be independent and uniformly distributed points inside the regular convex polygon with $2m$ sides, $m \in \mathbb{N}$, and unit area. We then have*

$$n^{1/2} \cdot \left(\frac{2}{\sqrt{m \sin(\pi/m)}} - D_n \right) \xrightarrow{\mathcal{D}} \min_{1 \leq k \leq m} \left(W_U^{(k)} + W_V^{(k)} \right)$$

as $n \to \infty$, where $W_U^{(k)}$, $W_V^{(k)}$, $k = 1, \ldots, m$, are independent Weibull distributed random variables with distribution function $G(t) = 1 - \exp\left(-t^2 \cot \frac{\pi}{2m}\right)$, $t > 0$.

Proof. By adopting the notations in Section 6.1, we have

$$F_Y^{(k)}(t) \sim \tan \frac{\beta}{2} \cdot t^2 = \cot \frac{\pi}{2m} \cdot t^2, \qquad t \to 0.$$

It follows that for each $t > 0$

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(n^{1/2} \cdot U_n^{(k)} \le t\right) &= 1 - \lim_{n\to\infty} \left(1 - F_Y^{(k)}(tn^{-1/2})\right)^n \\
&= 1 - \lim_{n\to\infty} \left(1 - \cot\frac{\pi}{2m} \cdot t^2 n^{-1}\right)^n \\
&= 1 - \exp\left(-\cot\frac{\pi}{2m} \cdot t^2\right) =: G(t).
\end{aligned}$$

Similarly, $n^{1/2} \cdot (s - V_n^{(k)})$ converges to the same Weibull law as $n \to \infty$. Hence,

$$n^{1/2} \cdot \left(s - M_n^{(k)}\right) \xrightarrow{\mathcal{D}} W_U^{(k)} + W_V^{(k)}$$

as $n \to \infty$ for $k = 1, \ldots, m$, where $W_U^{(k)}$ and $W_V^{(k)}$ are i.i.d. with the distribution function G. Now, we set $\varepsilon_n = tn^{-1/2}$ and prove that D_n and $\max_{1\le k\le m} M_n^{(k)}$ have the same asymptotic behavior as $n \to \infty$. Since $D_n \ge \max_{1\le k\le m} M_n^{(k)}$, one direction is trivial. For the other direction, suppose without loss of generality that $D_n = \|x_1 - x_2\| \ge s - \varepsilon_n$ holds. By the same geometric considerations as in the proof of Theorem 6.4, we obtain

$$\max_{1\le k\le m} M_n^{(k)} \ge \|x_1 - z_2\| = \sqrt{\|x_1 - x_2\|^2 - \|x_2 - z_2\|^2},$$

where z_2 denotes the orthogonal projection of x_2 onto the line parallel to the diagonal and passing through x_1 (see Figure 6.3). Since $\|x_2 - z_2\| \le \cot(\pi/2m) \cdot \varepsilon_n + C_1\varepsilon_n^2$ for some $C_1 > 0$ and sufficiently large n, a Taylor series expansion of ε_n at 0 yields

$$\begin{aligned}
\max_{1\le k\le m} M_n^{(k)} &\ge \sqrt{(s - \varepsilon_n)^2 - (\cot(\pi/2m) \cdot \varepsilon_n + C_1\varepsilon_n^2)^2} \\
&\ge s - \varepsilon_n - C_2\varepsilon_n^2,
\end{aligned}$$

where $C_2 > 0$ is some constant. Consequently, we have

$$\begin{aligned}
\mathbb{P}(D_n \ge s - \varepsilon_n) &\le \mathbb{P}\left(\max_{1\le k\le m} M_n^{(k)} \ge s - \varepsilon_n - C_2\varepsilon_n^2\right) \\
&= \mathbb{P}\left(n^{1/2}\left(s - \max_{1\le k\le m} M_n^{(k)}\right) \le t + C_2 t^2 n^{-1/2}\right) \\
&\to \mathbb{P}\left(\min_{1\le k\le m} \left(W_U^{(k)} + W_V^{(k)}\right) \le t\right)
\end{aligned}$$

as $n \to \infty$. This completes the proof. □

For $m = 3$ the support of the uniform distribution is a regular hexagon with unit area. The length of the major diagonals is $s = 2/\sqrt{3\sqrt{3}/2}$. Figure 6.16

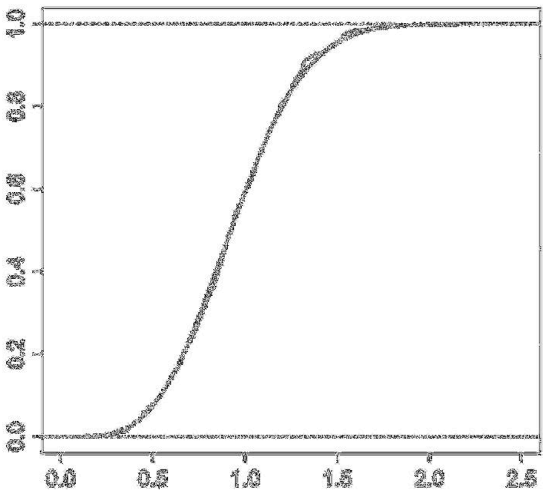

Figure 6.16: EDF of $n^{1/2} \cdot (2/\sqrt{3\sqrt{3}/2} - D_n)$ with $n = 1000$ and the limit distribution function for the largest distance in a regular hexagon (thick smooth curve).

shows an approximation of the limit law given in Theorem 6.8 by the EDF of $n^{1/2} \cdot (s - D_n)$, $n = 1000$.

Appel, Najim and Russo used in [4] a different method to approximate D_n and gave the same result for the asymptotic behavior of D_n in a regular convex polygon. They also provided some lower and upper bounds for the limit law of D_n in an ellipse with unit area. We shall derive in the following an sharper lower bound on the limit distribution function of suitable normalized D_n.

Theorem 6.9. *Let $X_1, X_2, \ldots$ be independent and uniformly distributed points in the ellipse with major half axis $a > 1/\sqrt{\pi}$ and unit area. We then have for each $t > 0$*

$$\liminf_n \mathbb{P}\left(n^{2/3} \cdot (2a - D_n) \leq t\right) \geq G * G(t),$$

where $G * G$ is the convolution of $G(t) = 1 - \exp\left(-\frac{4\sqrt{2}}{3\pi} a^{-3/2} t^{3/2}\right)$ with itself.

Proof. Since the ellipse has unit area, the minor half axis is $b = 1/(\pi a) < a$. Because of translation invariance we take without loss of generality

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (x, y) \in \mathbb{R}^2,$$

as the boundary function of the ellipse, which implies that $(0,0)$ and $(2a,0)$ are the endpoints of the major axis and $y = \pm\frac{1}{\pi a^2}\sqrt{2ax - x^2}$ for each (x,y) on the boundary. Adopt the notations of Section 6.1. Since the points are uniformly distributed, we obtain by integrating the boundary function

$$F_Y(t) = \frac{1}{2} - \frac{1}{\pi}\left[\frac{a-t}{a^2}\sqrt{2at - t^2} + \arcsin\left(1 - \frac{t}{a}\right)\right]$$

for each $t \in (0, 2a)$. It follows from a Taylor series expansion of F_Y around 0 that for each $t > 0$

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(n^{2/3} \cdot U_n \le t\right) &= 1 - \lim_{n\to\infty} \mathbb{P}\left(U_n > tn^{-2/3}\right) \\
&= 1 - \lim_{n\to\infty} \left(1 - F_Y(tn^{-2/3})\right)^n \\
&= 1 - \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{4\sqrt{2}}{3}a^{-3/2}t^{3/2}n^{-1}\right)\right)^n \\
&= 1 - \exp\left(-\frac{4\sqrt{2}}{3\pi}a^{-3/2}t^{3/2}\right) =: G(t).
\end{aligned}$$

By symmetry, $n^{2/3} \cdot (2a - V_n)$ converges to the same limit law as $n \to \infty$. Finally, since $D_n \ge M_n = V_n - U_n$, we obtain the lower bound given in Theorem 6.9. □

Figure 6.17 shows simulation results of the EDFs of $n^{2/3} \cdot (2a - D_n)$ with $a = 0.6$, $a = 0.8$, $a = 1$ and $a = 2$, and the bounds for the limit distribution function given in Theorem 6.9. In addition, the EDFs and the lower bounds lie between the lower and upper bounds by Appel, Najim and Russo [4], i.e., they stated that for $t > 0$

$$\begin{aligned}
\mathbb{P}\left(W_1 + W_2 \le \frac{t}{1+\gamma}\right) &\le \liminf_{n\to\infty} \mathbb{P}\left(n^{2/3} \cdot (2a - D_n) \le t\right) \\
&\le \limsup_{n\to\infty} \mathbb{P}\left(n^{2/3} \cdot (2a - D_n) \le t\right) \\
&\le \mathbb{P}\left(W_1 + W_2 \le t\right),
\end{aligned}$$

where $\gamma = 1/(\pi^2 a^4 - 1)$ and W_1, W_2 are i.i.d. with the distribution function $1 - \exp(-ct^{3/2})$ with $c = \frac{2}{3}(2\pi^2 a^4 - 1)\sqrt{\frac{2a}{(\pi^2 a^4 - 1)^3}}$. For small a, the upper and lower bounds differ considerably, and they become closer with growing a.

Turning back to the last crucial point stated at the beginning of this section, we note that D_n and the maximum of the sample ranges have the same asymptotic behavior, if the boundary function of the bounded support has a sub-$\sqrt{x}$ decay at the endpoints of the major axes. We state this requirement

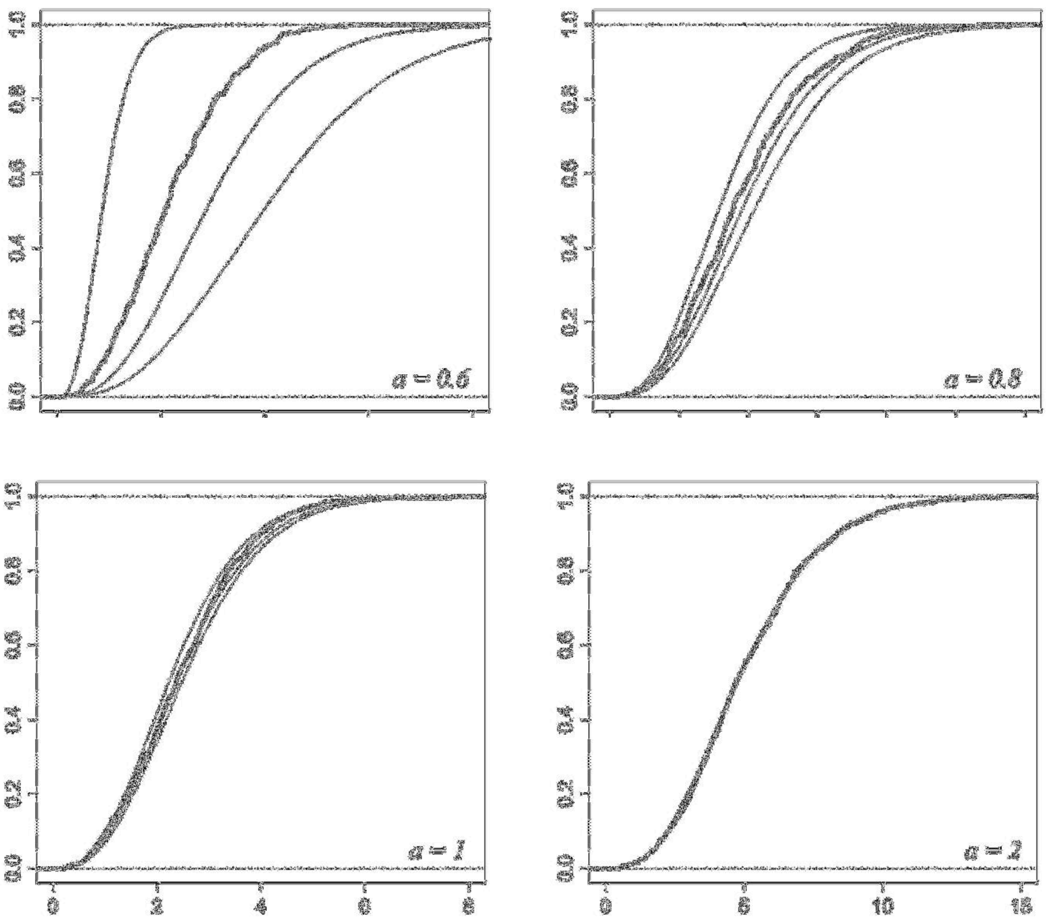

Figure 6.17: The thick curve in each image is the EDF of $n^{2/3} \cdot (2a - D_n)$, $n = 20000$, for different a. The smooth curves in each image are, from above to below, the upper bound given in [4], the lower bound given in Theorem 6.9 and the lower bound given in [4] for the limit distribution function.

more precisely by considering one of the major axes. Let $s > 0$ denote the length of the major axis, and its endpoints are $(0, 0)$ and $(s, 0)$. If the boundary function of the support is described by $h_1(x)$ and $h_2(x)$ with $h_1 \geq h_2$ on $[0, s]$ and $h_1(0) = h_2(0) = h_1(s) = h_2(s) = 0$, then they satisfy $h_1(x)/\sqrt{x} \to 0$, $h_2(x)/\sqrt{x} \to 0$ as $x \to 0$ and $h_1(x)/\sqrt{s - x} \to 0$, $h_2(x)/\sqrt{s - x} \to 0$ as $x \to s$. These conditions on the boundary function hold for the regular convex polygons. However, for the ellipses the boundary has a $\sqrt{x}$ decay, so that we derive only a lower bound of the limit law of D_n.

Chapter 7

Open problems and conjectures

At the end of this thesis, we want to highlight some open questions related to limit laws of extremes, and give some suggestions for further research.

7.1 Spherically symmetric distributions with unbounded support

In this thesis, the distribution of X_1 was assumed to have bounded support. If the support of $\mathbb{P}^{X_1}$ is unbounded, the largest interpoint distance D_n converges almost surely to infinity as the sample size increases. What is the possible limit law of D_n, when suitably rescaled, in this case?

To be more specific, suppose that $\mathbb{P}^{X_1}$ is spherically symmetric. Let $X_i = R_i U_i$, $i = 1, 2, \ldots$, be i.i.d. random points with independent radial component R_i and directional component U_i, where R_i has distribution function F with

$$\sup\{t > 0 : F(t) < 1\} = \infty,$$

and U_i has the uniform distribution on $\mathbb{S}^{d-1}$. If X_1 has a standard d-variate normal distribution, the limit law of D_n is a Gumbel distribution (see Matthews and Rukhin [30]). This result was generalized by Henze and Klein [23] to the case of the (short-tailed) spherically symmetric Kotz type distribution.

Henze and Lao [24] obtained the limit law of D_n, if F belongs to the maximum domain of attraction of the Fréchet distribution, i.e.,

$$1 - F(s) = s^{-\alpha} \cdot L(s), \qquad s > 0,$$

for some constant $\alpha > 0$ and some slowly varying function L. Since the right tail of F is regularly varying with index $-\alpha$, the technique used in [30] and [23] is not applicable. By using point process techniques, the limit law of D_n, when rescaled by $1/\inf\{t \in \mathbb{R} : F(t) \geq 1 - 1/n\}$, can be given as the supremum of distances between points of a certain Poisson process.

For other cases of spherically symmetric point distributions with unbounded support, the limit behavior of D_n is still unknown.

7.2 Radial distributions of exponential type

In the situation of Section 3.5 with a radial distribution of exponential type, we detected an interesting phenomenon. We found a threshold $2 - \varepsilon_n$, so that the expected number of exceedances $|X_i - X_j| > 2 - \varepsilon_n$ is stabilized, i.e. condition (3.1) is satisfied. But the variance of the random number tends to infinity as $n \to \infty$, i.e. condition (3.2) fails, so that the Poisson approximation theorem is not applicable. Moreover, it has been proved that this number converges to zero in probability. However, an honest limit law of D_n under this setting is still unknown.

7.3 Largest area of triangles formed by points in a circle

Suppose that, under the setting of Chapter 4, the points are uniformly distributed *inside* the unit circle. By some geometric considerations and tedious calculations, we deduce that

$$\begin{aligned} 1 - e^{-t^4/c_1} &\leq \liminf_{n \to \infty} \mathbb{P}\left(n^{3/4}\left(\frac{3\sqrt{3}}{4} - A_n\right) \leq t \right) \\ &\leq \limsup_{n \to \infty} \mathbb{P}\left(n^{3/4}\left(\frac{3\sqrt{3}}{4} - A_n\right) \leq t \right) \leq 1 - e^{-t^4/c_2} \end{aligned}$$

with $c_1 = 108\pi^2$ and $c_2 = \frac{81}{256}\pi^2$. We conjecture that $n^{3/4}\left(3\sqrt{3}/4 - A_n\right)$ has a nondegenerate Weibull limit distribution.

7.4 Largest distance between points in an ellipse

In Section 6.4 we gave a lower bound for the limit law of D_n, when the points are uniform inside an ellipse. The limit distribution of the maximum interpoint distance for points distributed uniformly in a (proper) ellipse is still unknown.

Appendix A

Proofs of Lemma 3.2 and 3.3 and some convergence theorems

Proof of Lemma 3.2. Since $\psi_1(s) \sim \psi_2(s)$ as $s \to 0$, it follows that for each $\delta > 0$ there is some $s_0 > 0$ such that

$$\left|\frac{\psi_1(s)}{\psi_2(s)} - 1\right| \leq \delta$$

and thus

$$(1-\delta)\cdot\psi_2(s) \leq \psi_1(s) \leq (1+\delta)\cdot\psi_2(s)$$

for all $s \in [0, s_0]$. For each $s \in [0, s_0]$ and $t \in [0, s]$ we therefore have

$$(1-\delta)\cdot\psi_2(t)\cdot h(s,t) \leq \psi_1(t)\cdot h(s,t) \leq (1+\delta)\cdot\psi_2(t)\cdot h(s,t),$$

and since $s - t \in [0, s_0]$ we also have

$$(1-\delta)\cdot\psi_2(s-t)\cdot h(s,t) \leq \psi_1(s-t)\cdot h(s,t) \leq (1+\delta)\cdot\psi_2(s-t)\cdot h(s,t).$$

By integrating we get

$$\begin{aligned}(1-\delta)\int_0^s \psi_2(t)\cdot h(s,t)\,dt &\leq \int_0^s \psi_1(t)\cdot h(s,t)\,dt \\ &\leq (1+\delta)\int_0^s \psi_2(t)\cdot h(s,t)\,dt\end{aligned}$$

and

$$\begin{aligned}(1-\delta)\int_0^s \psi_2(s-t)\cdot h(s,t)\,dt &\leq \int_0^s \psi_1(s-t)\cdot h(s,t)\,dt \\ &\leq (1+\delta)\int_0^s \psi_2(s-t)\cdot h(s,t)\,dt\end{aligned}$$

for all $s \in [0, s_0]$, and (a) and (b) follow immediately. □

Proof of Lemma 3.3. (a) follows immediately from Lemma 3.2. To prove (b), note that by Fubini′s theorem, we have

$$F * F(s) \;=\; \int_0^s F(s-x)\,dF(x) \;=\; \int_0^s F(s-x)F'(x)\,dx, \qquad s \geq 0.$$

Since $F'(s) \sim \Psi'(s)$ and $F(s) \sim \Psi(s)$ as $s \to 0$, for each $\delta > 0$ there is some $s_0 > 0$ such that

$$\left|\frac{F(s-x)F'(x)}{\Psi(s-x)\Psi'(x)} - 1\right| \leq \delta$$

for each $s \in (0, s_0)$ and each $x \in (0, s)$, whence

$$(1-\delta)\Psi(s-x)\Psi'(x) \leq F(s-x)F'(x) \leq (1+\delta)\Psi(s-x)\Psi'(x)$$

for each $s \in (0, s_0)$ and each $x \in (0, s)$. By integrating we get

$$\begin{aligned}(1-\delta)\int_0^s \Psi(s-x)\Psi'(x)\,dx \;&\leq\; \int_0^s F(s-x)F'(x)\,dx \\ &\leq\; (1+\delta)\int_0^s \Psi(s-x)\Psi'(x)\,dx\end{aligned}$$

for each $s \in (0, s_0)$, which proves the assertion. □

The following convergence theorems are useful in the proof of Lemma 3.4:

Theorem A.1 (Generalized Lebesgue′s differentiation theorem)**.** *Let ν be a Borel regular measure over $\mathbb{R}^d$ such that every bounded subset of $\mathbb{R}^d$ has finite ν measure, and let f be an $\mathbb{R}^d$-valued ν-measurable function such that $\int_A \|f\|\,d\nu < \infty$ for every bounded ν-measurable subset $A \subset \mathbb{R}^d$. Then, for ν-almost every $x \in \mathbb{R}^d$*

$$\lim_{n\to\infty} \frac{1}{\nu(B_n)} \int_{B_n} f\,d\nu \;=\; f(x),$$

where $(B_n)_{n\geq 1}$ denotes a sequence of balls centered at x, and the diameter of B_n tends to 0 as $n \to \infty$.

We can also replace the balls (B_n) by a family of sets U, which satisfies the following conditions: Firstly, there is a constant $c > 0$ such that each set U from the family is contained in a ball B_n with $\nu(U) \geq c\nu(B_n)$; secondly, each $x \in \mathbb{R}^d$ is contained in arbitrarily small sets from the family. Then, the same result holds when these sets shrink to x.

Theorem A.2 (Dominated convergence theorem)**.** *Let $f_1, f_2, \ldots$ be a sequence of real-valued measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$. Suppose that f_n f pointwise μ-almost everywhere as n , and that f_n $\leq g$ for all n, where g is a μ-integrable nonnegative function on Ω. Then f is μ-integrable, and*

$$f\,d\mu = \lim_{n} f_n\,d\mu.$$

Appendix B

Hausdorff measure and surface area measure

B.1 Hausdorff measure

Hausdorff measure is a type of outer measure, which measures some "very small" subsets of $\mathbb{R}^n$. This brief introduction refers to the book by Evans and Gariepy [12].

Let (X, d) be a metric space. For any $A \subset X$ we define the diameter of A as

$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y), \qquad \operatorname{diam}(\emptyset) := 0.$$

Let $s \geq 0$ and $\delta > 0$. Define

$$\mathscr{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} \omega_s \left(\frac{\operatorname{diam}(B_i)}{2} \right)^s : A \subset \bigcup_{i=1}^{\infty} B_i,\ \operatorname{diam}(B_i) \leq \delta \right\},$$

where the infimum is taken over all countable covers of A by sets $B_i \subset X$, $i \in \mathbb{N}$, satisfying $\operatorname{diam}(B_i) \leq \delta$, and

$$\omega_s = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2} + 1\right)}$$

is the volume of the s-dimensional unit ball.

Notice that $\mathscr{H}^s_\delta(A)$ is decreasing in δ, since for $\delta' > \delta$ the corresponding family $(B_i)_{i \in \mathbb{N}}$ contains more sets and hence the infimum is smaller. Thus, the limit $\lim_{\delta \to 0} \mathscr{H}^s_\delta(A)$ exists, and the s-dimensional Hausdorff measure of A is defined as

$$\mathscr{H}^s(A) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(A) = \sup_{\delta > 0} \mathscr{H}^s_\delta(A).$$

Example B.1. (i) If $A \subset \mathbb{R}^n$, then $\mathscr{H}^0(A)$ counts the number of points in A, so it is the so-called counting measure.

(ii) If A is a simple curve in $\mathbb{R}^n$, then $\mathscr{H}^1(A)$ is the length of the curve A.

(iii) If $A \subset \mathbb{R}^2$ measurable, then $\mathscr{H}^2(A)$ measures the area of A.

Moreover, the n-dimensional Lebesgue measure and n-dimensional Hausdorff measure agree on $\mathbb{R}^n$. A detailed proof can be found in Evans and Gariepy [12], Section 2.2.

B.2 Surface area measure of a spherical cap

For a point $u \in \mathbb{S}^{d-1}$ and a $\eta \geq 0$, let

$$C_u(\eta) = \left\{ v \in \mathbb{S}^{d-1} : \measuredangle(-u, v) \leq \sqrt{\eta} \right\}$$

be the spherical cap centered at $-u$ with the maximal central angle $\sqrt{\eta}$. One can obtain the surface area of $C_u(\eta)$ by integrating in spherical coordinates. It is very similar to the computation of the surface area of a sphere, which can be found in many literatures, e.g. Walter [39] (page 253-254) or Amann and Escher [3] (page 198-201).

Now, we derive step by step the following asymptotic result as $\eta \to 0$:

$$\begin{aligned} \mu^{d-1}(C_u(\eta)) \;&\sim\; \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot (d-1) \cdot \int_{t=0}^{\sqrt{\eta}} t^{d-2}\,dt \\ &= \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot \eta^{\frac{d-1}{2}}, \end{aligned} \tag{B.1}$$

which is applied in the proof of Lemma 3.4.

By symmetry, we choose $-u = (0, 0, \ldots, 0, 1)$ without loss of generality. Then

$$C_u(\eta) = \{(y_1, \ldots, y_{d-1}, H(y_1, \ldots, y_{d-1})) \in \mathbb{R}^d : y_1^2 + \cdots + y_{d-1}^2 \leq \sin^2 \sqrt{\eta}\}$$

with

$$H(y_1, \ldots, y_{d-1}) := \sqrt{1 - y_1^2 - \cdots - y_{d-1}^2}$$

Thus, the surface area can be derived by the method stated in [39] (§ 8.9). Put $y := (y_1, \dots, y_{d-1}) \in \mathbb{R}^{d-1}$ and $A := \{ y \in \mathbb{R}^{d-1} : \|y\| \leq \sin \bar{\eta} \}$. Since

$$\frac{\partial H}{\partial y_i} = \frac{-y_i}{\sqrt{1 - \|y\|^2}}$$

for $i = 1, \dots, d-1$, we have

$$\begin{aligned}
\mu^{d-1}(C_u(\eta)) &= \int_A \sqrt{1 + \left(\frac{\partial H}{\partial y_1}\right)^2 + \cdots + \left(\frac{\partial H}{\partial y_{d-1}}\right)^2}\, dy \\
&= \int_A \frac{1}{\sqrt{1 - \|y\|^2}}\, dy.
\end{aligned}$$

To proceed the computation, we transform the integral to polar coordinates. Set

$$\begin{aligned}
y_1 &= r \sin\theta_1 \ \dots\ \dots\ \dots\ \sin\theta_{d-3} \sin\theta_{d-2} \\
y_2 &= r \sin\theta_1 \ \dots\ \dots\ \dots\ \sin\theta_{d-3} \cos\theta_{d-2} \\
y_3 &= r \sin\theta_1 \ \dots\ \sin\theta_{d-4} \cos\theta_{d-3} \\
&\ \vdots \\
y_{d-2} &= r \sin\theta_1 \cos\theta_2, \\
y_{d-1} &= r \cos\theta_1,
\end{aligned}$$

where $r \geq 0$, $0 \leq \theta_{d-2} \leq 2\pi$ and $0 \leq \theta_i \leq \pi$ for $i = 1, \dots, d-3$. The transformation yields

$$\begin{aligned}
\mu^{d-1}(C_u(\eta)) = & \int_{r=0}^{\sin\bar{\eta}} \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{d-3}=0}^{\pi} \int_{\theta_{d-2}=0}^{2\pi} \frac{1}{\sqrt{1 - r^2}} \\
& r^{d-2} (\sin\theta_1)^{d-3} (\sin\theta_2)^{d-4} \dots \sin\theta_{d-3}\, d\theta_{d-2} \dots d\theta_1\, dr,
\end{aligned}$$

where the $(d-2)$-dimensional integral of the angles $\theta_1, \dots, \theta_{d-2}$ is known as

$$\begin{aligned}
& \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{d-3}=0}^{\pi} \int_{\theta_{d-2}=0}^{2\pi} (\sin\theta_1)^{d-3} (\sin\theta_2)^{d-4} \dots \sin\theta_{d-3}\, d\theta_{d-2} \dots d\theta_1 \\
= & \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} = \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot (d-1)
\end{aligned}$$

(see e.g. [39] p. 254). Thus, by substituting $t = \arcsin r$, the surface area of

$C_u(\eta)$ reduces to

$$\begin{aligned}
\mu^{d-1}(C_u(\eta)) &= \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot (d-1) \cdot \int_{r=0}^{\sin\sqrt{\eta}} \frac{r^{d-2}}{\sqrt{1-r^2}}\, dr \\
&= \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot (d-1) \cdot \int_{t=0}^{\sqrt{\eta}} (\sin t)^{d-2} dt.
\end{aligned}$$

Recall that we observe here the surface area for a very small η. Applying the Taylor series expansion of $\sin t$ at 0, Lemma 3.2 leads to (B.1) as $\eta \to 0$.

Appendix C

Regularly varying functions

Nowadays, the theory of regularly varying functions, imported by Feller [14] into probability theory, plays a crucial role in extreme value theory, especially for questions about domains of attraction. We state here some useful results for our concept. Further information can be found e.g. in Embrechts et al. [11] or Resnick [36].

Definition C.1. (a) A positive measurable function h on $(0, \infty)$ is said to be regularly varying at 0 of index $\alpha \in \mathbb{R}$ (we write $h \in \mathcal{R}_\alpha$) if

$$\lim_{s \to 0} \frac{h(st)}{h(s)} = t^\alpha, \qquad t > 0.$$

(b) A positive measurable function L on $(0, \infty)$ is said to be slowly varying at 0 (we write $L \in \mathcal{R}_0$) if

$$\lim_{s \to 0} \frac{L(st)}{L(s)} = 1, \qquad t > 0.$$

(c) A positive measurable function h on $(0, \infty)$ is said to be rapidly varying with index ∞ (we write $h \in \mathcal{R}_\infty$) if

$$\lim_{s \to 0} \frac{h(st)}{h(s)} = \begin{cases} 0, & \text{if } 0 < t < 1, \\ \infty, & \text{if } t > 1. \end{cases}$$

If $h \in \mathcal{R}_\alpha$ then $h(t)/t^\alpha \in \mathcal{R}_0$, so a α-varying function is always representable as $t^\alpha L(t)$. Note that the convergences above are uniform on each compact subset of $(0, \infty)$. Moreover, we can represent a regularly varying function $h \in \mathcal{R}_\alpha$ for some $\alpha \in \mathbb{R}$ as follows:

$$h(x) = c(x) \exp\left(\int_z^x \frac{\delta(u)}{u}\, du \right), \qquad x \geq z,$$

for some $z > 0$, where $c(\cdot)$ and $\delta(\cdot)$ are measurable functions with $c(x) \to c_0 \in (0,\infty)$ and $\delta(x) \to \alpha$ as $x \to 0$. This representation is often applicable.

The following theorem, which can be found in [11] (Theorem A.3.7), gives the asymptotic behavior of the differentiation of regularly varying functions.

Theorem C.1 (Monotone density theorem)**.** *Let*

$$F(x) = \int_0^x f(y)\,dy \qquad \left(\text{or } \int_x^\infty f(y)\,dy\right),$$

where f is monotone on (z,∞) for some $z > 0$. If

$$F(x) \sim c x^{\alpha} L(x), \qquad x \to 0,$$

with $c \geq 0$, $\alpha \in \mathbb{R}$ and $L \in \mathcal{R}_0$, then

$$f(x) \sim c\alpha x^{\alpha-1} L(x), \qquad x \to 0.$$

Appendix D

The computation of the limit laws in Chapter 5

D.1 The limit law in Section 5.1

Let $E_1^{(1)}, E_m^{(2)}, E_{m+1}^{(1)}, E_{2m}^{(2)}$ be i.i.d. random variables with distribution function $F(t) := 1 - \exp\left(-\frac{2t}{a}\right)$ and density $f(t) := \frac{2}{a}\exp\left(-\frac{2t}{a}\right)$ for $t \geq 0$, where a is a positive constant. Define

$$E := \min\left(E_1^{(1)} + E_m^{(2)},\ E_1^{(1)} + E_{m+1}^{(1)},\ E_{m+1}^{(1)} + E_{2m}^{(2)},\ E_m^{(2)} + E_{2m}^{(2)} \right).$$

In the following we derive the distribution function of E. For each $t \geq 0$ we have

$$\begin{aligned}
&\mathbb{P}\left(E > t\right) \\
= \ &\mathbb{P}\left(E_1^{(1)} + E_m^{(2)} > t, E_1^{(1)} + E_{m+1}^{(1)} > t, E_{m+1}^{(1)} + E_{2m}^{(2)} > t, E_m^{(2)} + E_{2m}^{(2)} > t\right) \\
= \ &\int_{u=0}\int_{v=0} \mathbb{P}\left(u + E_m^{(2)} > t,\ u + E_{m+1}^{(1)} > t,\ v + E_{m+1}^{(1)} > t,\ v + E_m^{(2)} > t\right) \\
&\qquad\qquad d\mathbb{P}^{E_{2m}^{(2)}}(v)\, d\mathbb{P}^{E_1^{(1)}}(u) \\
= \ &\int_{u=0}\int_{v=0} \mathbb{P}\left(E_m^{(2)} > t - u,\ E_m^{(2)} > t - v\right)^2 \cdot f(v)\, dv \cdot f(u)\, du. \qquad \text{(D.1)}
\end{aligned}$$

We compute the double integral by splitting up the region of integration into four disjoint cases as follows:

1.) $0 \leq u \leq t$ and $0 \leq v \leq u$:

$$
\begin{aligned}
& \int_{u=0}^{t}\int_{v=0}^{u} \mathbb{P}\left(E_m^{(2)} > t-u,\ E_m^{(2)} > t-v\right)^2 \cdot f(v)\,dv \cdot f(u)\,du \\
= & \int_{u=0}^{t}\int_{v=0}^{u} \mathbb{P}\left(E_3 > t-v\right)^2 \cdot f(v)\,dv \cdot f(u)\,du \\
= & \int_{u=0}^{t}\int_{v=0}^{u} e^{-\frac{4}{a}(t-\ v)} \cdot \frac{2}{a}e^{-\frac{2v}{a}}\,dv \cdot \frac{2}{a}e^{-\frac{2u}{a}}\,du \\
= & \ e^{-\frac{4t}{a}} \int_{u=0}^{t} \frac{2}{a} - \frac{2}{a}e^{-\frac{2u}{a}} \ du \\
= & \ \frac{2t}{a}e^{-\frac{4t}{a}} + e^{-\frac{6t}{a}} - e^{-\frac{4t}{a}},
\end{aligned}
$$

2.) $0 \le u \le t$ and $u \le v \le$:

$$
\begin{aligned}
& \int_{u=0}^{t}\int_{v=u} \mathbb{P}\left(E_m^{(2)} > t-u,\ E_m^{(2)} > t-v\right)^2 \ f(v)\,dv \cdot f(u)\,du \\
= & \int_{u=0}^{t}\int_{v=u} \mathbb{P}\left(E_m^{(2)} > t-u\right)^2 \cdot f(v)\,dv \cdot f(u)\,du \\
= & \int_{u=0}^{t}\int_{v=u} e^{-\frac{4}{a}(t-\ u)} \cdot \frac{2}{a}e^{-\frac{2v}{a}}\,dv \cdot \frac{2}{a}e^{-\frac{2u}{a}}\,du \\
= & \ e^{-\frac{4t}{a}} \int_{u=0}^{t} \frac{2}{a}\,du \\
= & \ \frac{2t}{a}e^{-\frac{4t}{a}},
\end{aligned}
$$

3.) $t \le u \le$ and $0 \le v \le t$:

$$
\begin{aligned}
& \int_{u=t}\int_{v=0}^{t} \mathbb{P}\left(E_m^{(2)} > t-u,\ E_m^{(2)} > t-v\right)^2 \cdot f(v)\,dv \cdot f(u)\,du \\
= & \int_{u=t}\int_{v=0}^{t} \mathbb{P}\left(E_m^{(2)} > t-v\right)^2 \cdot f(v)\,dv \cdot f(u)\,du \\
= & \int_{u=t}\int_{v=0}^{t} e^{-\frac{4}{a}(t-\ v)} \cdot \frac{2}{a}e^{-\frac{2v}{a}}\,dv \cdot \frac{2}{a}e^{-\frac{2u}{a}}\,du \\
= & \ e^{-\frac{4t}{a}}\left(e^{\frac{2t}{a}} - 1\right) \int_{u=t} \frac{2}{a}e^{-\frac{2u}{a}}\,du \\
= & \ e^{-\frac{4t}{a}} - e^{-\frac{6t}{a}},
\end{aligned}
$$

4.) $t \le u \le$ and $t \le v \le$:

$$
\begin{aligned}
& \int_{u=t}\int_{v=t} \mathbb{P}\left(E_m^{(2)} > t-u,\ E_m^{(2)} > t-v\right)^2 \cdot f(v)\,dv \cdot f(u)\,du \\
=& \int_{u=t}\int_{v=t} \mathbb{P}\left(E_m^{(2)} > 0\right)^2 \cdot f(v)\,dv \cdot f(u)\,du \\
=& \int_{u=t}\int_{v=t} \frac{2}{a}e^{-\frac{2v}{a}}\,dv \cdot \frac{2}{a}e^{-\frac{2u}{a}}\,du \\
=& \; e^{-\frac{2t}{a}} \int_{u=t} \frac{2}{a}e^{-\frac{2u}{a}}\,du \\
=& \; e^{-\frac{4t}{a}}.
\end{aligned}
$$

Plugging these four parts of double integrals into (D.1), we obtain

$$
\mathbb{P}(E \le t) \;=\; 1 - \mathbb{P}(E > t) = 1 - \left(\frac{4t}{a} + 1\right) \cdot e^{-\frac{4t}{a}}.
$$

D.2 The limit law in Section 5.2

Let $E_1, \ldots, E_6$ be independent and $Exp(2/\sqrt{3})$-distributed random variables with distribution function $F(t) = 1 - \exp\left(-\frac{2}{\sqrt{3}}t\right)$ and density $f(t) = \frac{2}{\sqrt{3}} \cdot \exp\left(-\frac{2}{\sqrt{3}}t\right)$ for $t \ge 0$. The limit law E is presented as the minimum given in (5.11), whence for $t \ge 0$

$$
\begin{aligned}
\mathbb{P}(E > t) = \mathbb{P}\Big(& E_1 + E_2 > t,\ E_1 + \tfrac{1}{2}E_3 > t,\ \tfrac{1}{2}E_1 + \tfrac{1}{2}E_4 > t,\ \tfrac{1}{2}E_1 + E_5 > t, \\
& \tfrac{1}{2}E_2 + E_4 > t,\ \tfrac{1}{2}E_2 + \tfrac{1}{2}E_5 > t,\ E_2 + \tfrac{1}{2}E_6 > t,\ E_3 + E_4 > t, \\
& E_3 + \tfrac{1}{2}E_5 > t,\ \tfrac{1}{2}E_3 + \tfrac{1}{2}E_6 > t,\ \tfrac{1}{2}E_4 + E_6 > t,\ E_5 + E_6 > t\Big).
\end{aligned}
$$

We de ne the following events:

$$
\begin{aligned}
A_1 &= \left\{E_1 + E_2 > t,\ E_1 + \tfrac{1}{2}E_3 > t,\ \tfrac{1}{2}E_1 + \tfrac{1}{2}E_4 > t,\ \tfrac{1}{2}E_1 + E_5 > t\right\}, \\
A_2 &= \left\{\tfrac{1}{2}E_2 + E_4 > t,\ \tfrac{1}{2}E_2 + \tfrac{1}{2}E_5 > t,\ E_2 + \tfrac{1}{2}E_6 > t\right\}, \\
A_3 &= \left\{E_3 + E_4 > t,\ E_3 + \tfrac{1}{2}E_5 > t,\ \tfrac{1}{2}E_3 + \tfrac{1}{2}E_6 > t\right\}, \\
A_4 &= \left\{\tfrac{1}{2}E_4 + E_6 > t\right\}, \\
A_5 &= \left\{E_5 + E_6 > t\right\}.
\end{aligned}
$$

By conditioning on $E_1 = z$ with $z > 0$, we can split the probability $\mathbb{P}(E > t)$

into three parts $\mathbb{P}(E > t) = I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= \int_{z=0}^{t} \mathbb{P}\Big(E_2 > t - z,\ E_3 > 2t - 2z,\ E_4 > 2t - z,\ E_5 > t - \tfrac{z}{2} \\
&\qquad \setminus A_2 \setminus A_3 \setminus A_4 \setminus A_5) f(z)\, dz, \\
I_2 &= \int_{z=t}^{2t} \mathbb{P}\Big(E_4 > 2t - z,\ E_5 > t - \tfrac{z}{2} \setminus A_2 \setminus A_3 \setminus A_4 \setminus A_5\Big) f(z)\, dz, \\
I_3 &= \int_{z=2t} \mathbb{P}(A_2 \setminus A_3 \setminus A_4 \setminus A_5)\, f(z)\, dz.
\end{aligned}$$

We can further split each of the three integrals by conditioning on $E_2 = w$ for $w > 0$. For example, we have $I_3 = I_{3,1} + I_{3,2} + I_{3,3}$, where

$$\begin{aligned}
I_{3,1} &= \int_{z=2t}\int_{w=0}^{t} \mathbb{P}(E_4 > t - \tfrac{w}{2}, E_5 > 2t - w, E_6 > 2t - 2w \\
&\qquad \setminus A_3 \setminus A_4 \setminus A_5) f(w) f(z)\, dw\, dz, \\
I_{3,2} &= \int_{z=2t}\int_{w=t}^{2t} \mathbb{P}\Big(E_4 > t - \tfrac{w}{2}, E_5 > 2t - w \setminus A_3 \setminus A_4 \setminus A_5\Big) f(w) f(z)\, dw\, dz, \\
I_{3,3} &= \int_{z=2t}\int_{w=2t} \mathbb{P}(A_3 \setminus A_4 \setminus A_5)\, f(w) f(z)\, dw\, dz.
\end{aligned}$$

Then, conditioning on $E_3 = v$ for $v > 0$ yields a further splitting for each of $I_{3,1}$, $I_{3,2}$ and $I_{3,3}$. For example, we have $I_{3,3} = I_{3,3,1} + I_{3,3,2} + I_{3,3,3}$ with

$$\begin{aligned}
I_{3,3,1} &= \int_{z=2t}\int_{w=2t}\int_{v=0}^{t} \mathbb{P}(E_4 > t - v, E_5 > 2t - 2v, E_6 > 2t - v \\
&\qquad \setminus A_4 \setminus A_5) f(v) f(w) f(z)\, dv\, dw\, dz, \\
I_{3,3,2} &= \int_{z=2t}\int_{w=2t}\int_{v=t}^{2t} \mathbb{P}(E_6 > 2t - v \setminus A_4 \setminus A_5) f(v) f(w) f(z)\, dv\, dw\, dz, \\
I_{3,3,3} &= \int_{z=2t}\int_{w=2t}\int_{v=2t} \mathbb{P}(A_4 \setminus A_5)\, f(v) f(w) f(z)\, dv\, dw\, dz.
\end{aligned}$$

In the last step, we compute the integrals by conditioning on $E_6 = u$ for $u > 0$. Firstly, we have

$$
\begin{aligned}
I_{3,3,1} &= \int_{z=2t}\int_{w=2t}\int_{v=0}^{t}\int_{u=2t-v} \mathbb{P}\left(E_4 > t-v, E_5 > 2t-2v\right) f(u)f(v)f(w)f(z) \\
&\qquad du\, dv\, dw\, dz \\
&= \int_{z=2t}\int_{w=2t}\int_{v=0}^{t}\int_{u=2t-v} e^{-\frac{2(t-v)}{\sqrt{3}}} e^{-\frac{2(2t-2v)}{\sqrt{3}}} \left(\frac{2}{\sqrt{3}}\right)^4 e^{-\frac{2u}{\sqrt{3}}} e^{-\frac{2v}{\sqrt{3}}} e^{-\frac{2w}{\sqrt{3}}} e^{-\frac{2z}{\sqrt{3}}} \\
&\qquad du\, dv\, dw\, dz \\
&= \frac{1}{3}\left(e^{2\sqrt{3}t} - 1\right) e^{-6\sqrt{3}t}.
\end{aligned}
$$

Secondly, it follows that

$$
\begin{aligned}
I_{3,3,2} &= \int_{z=2t}\int_{w=2t}\int_{v=t}^{2t}\int_{u=2t-v} \mathbb{P}\left(E_4 > 2t-2u, E_5 > t-u\right) f(u)f(v)f(w)f(z) \\
&\qquad du\, dv\, dw\, dz \\
&= \int_{z=2t}\int_{w=2t}\int_{v=t}^{2t}\int_{u=2t-v}^{t} \mathbb{P}\left(E_4 > 2t-2u, E_5 > t-u\right) f(u)f(v)f(w)f(z) \\
&\qquad du\, dv\, dw\, dz \\
&\quad + \int_{z=2t}\int_{w=2t}\int_{v=t}^{2t}\int_{u=t} f(u)f(v)f(w)f(z)\, du\, dv\, dw\, dz \\
&= \frac{4}{3}e^{-4\sqrt{3}t} + \frac{1}{6}e^{-6\sqrt{3}t} - \frac{3}{2}e^{-\frac{14}{3}t}.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
I_{3,3,3} &= \int_{z=2t}\int_{w=2t}\int_{v=2t}\int_{u=0} \mathbb{P}\left(E_4 > 2t-2u, E_5 > t-u\right) f(u)f(v)f(w)f(z) \\
&\qquad du\, dv\, dw\, dz \\
&= \int_{z=2t}\int_{w=2t}\int_{v=2t}\int_{u=0}^{t} \mathbb{P}\left(E_4 > 2t-2u, E_5 > t-u\right) f(u)f(v)f(w)f(z) \\
&\qquad du\, dv\, dw\, dz \\
&\quad + \int_{z=2t}\int_{w=2t}\int_{v=2t}\int_{u=t} f(u)f(v)f(w)f(z)\, du\, dv\, dw\, dz \\
&= \frac{3}{2}e^{-\frac{14}{3}t} - \frac{1}{2}e^{-6\sqrt{3}t}.
\end{aligned}
$$

All the other split integrals are computed by a Maple program. Finally, the probability $\mathbb{P}(E > t)$ can be obtained by summing up all of the integrals, and the limit distribution function is equal to $1 - \mathbb{P}(E > t)$.

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This work establishes several weak limit laws for problems in geometric extreme value theory. We find the limit law of the maximum Euclidean distance of independent and identically distributed points, as the number of points tends to infinity, under certain assumptions on the underlying distribution. For points in a ball, a main tool of proof is a Poisson approximation theorem. This method is also applicable for some other functionals, such as the maximum area or the maximum perimeter of triangles formed by point triplets. For points distributed inside a cube, inside a polygon or on the edges of a polygon, the limit distribution of the largest interpoint distance is obtained by classical extreme value theory and some geometric considerations.

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